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The Process of Probability Problem Solving: Use of External Visual Representations

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We investigate the role of external inscriptions, particularly those of a spatial or visual nature, in the solution of probability word problems. We define a taxonomy of external visual representations used in probability problem solving that includes *pictures, spatial reorganization of the given information, outcome listings, contingency tables, Venn diagrams, trees,* and *novel graphical representations.* We also propose a process model for probability problem solving (PPS) and use it as a framework to better understand how and why external visual representations are used. In a study of 34 novice probability problem solvers, participants worked to solve six probability word problems covering six probability subtopics. Both written and verbal structured interview protocols were analyzed to investigate when and how external visual representations are spontaneously used by problem solvers. Analyses of the coded transcripts showed that participants' probability problem-solving efforts move through the stages of PPS in a sequential but not always linear manner, sometimes exhibiting iterated attempts to represent the problem mathematically and to find a solution strategy. Results showed that use of specific external visual representations was associated with specific probability topics, and that certain choices of representation are associated with higher rates of solution success. These findings suggest that an external visual representation can facilitate probability problem solving, but only when an appropriate representation is chosen. Finally, we present evidence to show that external visual representations are usually created and first used during the stages of representing the problem mathematically and finding a solution strategy. However, pictures are often created during the initial stage of problem text understanding, and tables are sometimes created during computation of the solution.

Visualization has long been thought to play an important role in mathematics problem solving (e.g., Hadamard, 1945). When mathematical problems are especially difficult, or when solutions must be shared with others, problem solvers may externalize these visualizations by making inscriptions on paper or other media (e.g., Clement, Lochhead, & Monk, 1981; Corter & Zahner, 2007; Latour & Woolgar, 1986; Roth & McGinn, 1998; Russell, 2000; Schreiber, 2004). In

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some areas of mathematics, such as geometry, understanding and use of pictures and diagrams is considered to be an integral part of the domain knowledge (National Council of Teachers of Mathematics, 2003). In other areas of mathematics, external visualizations may not be an inherent part of the domain knowledge, but may still be frequently used as a means of solving problems or to pursue mathematical discovery (e.g., English, 1997; Polya, 1957; Presmeg, 2006).

Probability is one particular area in which mathematics problem solvers may rely on both internal visualizations and external sketches and diagrams. To illustrate, consider the following problem, a version of which circulated a few years ago on an Internet discussion list concerned with probability.

Six pieces of rope are threaded through a cardboard tube open at both ends. You randomly select two of the rope ends sticking out of one end of the tube, and tie them together. You repeat this action with two of the remaining rope ends, then again with the last two ends. At the other end of the tube you go through the same procedure. What is the probability that you have created one huge loop of rope?

Problem solvers trying to read and solve this problem often report visualizing the tube and the ropes and mentally manipulating these images to envision the knots being tied. Sometimes external sketches or diagrams may be created by a problem solver to aid in understanding the problem text (Corter & Zahner, 2007). Visualization can also aid the problem solver at later stages when mathematical and probabilistic concepts are applied to find the requested probability. At these later stages, the problem solver may again resort to using external inscriptions (e.g., diagrams, formulas, arithmetic calculations) to ease requirements on working memory or to otherwise facilitate the solution process. In this paper we investigate the spontaneous (unprompted) use of external inscriptions by problem solvers attempting probability word problems and relate the use of these external inscriptions to specific types of problems and to specific stages of the problem solving process.

Visual Representations and Problem Solving

Although mental imagery is widely believed to play a role in mathematical reasoning (e.g., Hegarty & Kozhevnikov, 1999; Lean & Clements, 1981; Polya, 1957; Presmeg, 1986, 2006; Rival, 1987; Sfard, 1994), surprisingly little empirical evidence exists to support this idea (Campbell, Collis, & Watson, 1995; Douville & Pugalee, 2003). Perhaps this is because of the difficulty in studying mental imagery in a rigorous way. In contrast, considerable evidence has accrued to show that external visual representations, especially diagrams, can facilitate problem solving in logic, science, and mathematics (e.g., Kaufmann, 1990; Molitor, Ballstaedt, & Mandl, 1989; Santos-Trigo, 1996; Schwartz & Martin, 2004; Stylianou & Silver, 2004; Tversky, 2001).

In studying external representations, we find it useful to distinguish between two types of studies on the use of external visualizations for problem solving. The first type of study, by far the most common, examines the effects on problem-solving activities and on success of diagrams provided by the experimenter or instructor. The visual representations investigated in these studies ranged from diagrams that accompanied text describing systems (e.g., Hall, Bailey, & Tillman, 1997; Mayer, 1989; Mayer & Gallini, 1990) to actual physical models of scientific systems (e.g., Penner, Giles, Lehrer, & Schauble, 1996). A common finding of these studies was that experimenter-provided external visuals facilitate problem-solving success. Many of the studies also conclude that external visual representations can aid in the development of student understanding of physical systems and mechanisms.

The second type of study examines the effects of student-generated diagrams on problem solving. Several previous studies have examined the effects of asking problem solvers to generate their own diagrams (e.g., Diezmann, 1995; Lehrer, Schauble, Carpenter, & Penner, 2000; Schwartz & Martin, 2004; Stylianou, 2002) while engaged in problem solving, usually finding a facilitative effect. In contrast, our interests are primarily in studying the spontaneous or unprompted creation of external visual representations by students engaged in probability problem solving (e.g., Corter & Zahner, 2007; Zahner & Corter, 2002). Some recent studies on the spontaneous use of external visual representations in mathematics problem solving (e.g., de Hevia & Spelke, 2009; Edens & Potter, 2008; Uesaka, Manolo, & Ichikawa, 2007) suggest that external representations are facilitative in the problem-solving process, although these studies have not focused on problem-solving in probability.

We are particularly interested in probability problem solving, and in the use of external representations that convey information in spatial terms, such as diagrams, pictures, and spatially organized displays, rather than in external inscriptions that use formal or natural language, such as formulas and calculations. Note that use of diagrams is generally acknowledged to be an integral part of mathematical knowledge in topics such as geometry or functions (e.g., Koedinger & Anderson, 1997; Nemirovsky, 1994; Sedlmeier & Gigerenzer, 2001). A number of studies have found that the use of “Geometer’s Sketchpad,” a geometry graphing computer program, can be helpful in developing students’ concepts and problem solving in geometry (Hannafin, Burruss, & Little, 2001; Hannafin & Scott, 1998; Hollebrands, 2003; Weaver & Quinn, 1999). These researchers have suggested that the tool is useful because it makes the geometric diagrams the central focus of the problem-solving process and allows students to explore the diagrams dynamically, altering points, lines, and arcs (Olive, 1998). In other mathematical topics, such as probability problem solving (PPS), the use of visual representations is not always considered to be an inherent part of the target domain knowledge; rather it may be considered more as a general technique in the mathematician’s toolbox (e.g., Polya, 1957). Only a few prior studies have directly addressed the role of visual strategies in PPS, thus the use of diagrams and other external visual devices in this domain is still not thoroughly understood.

Types of External Visual Representations

In the present paper we attempt to identify the stages of processing that students go through in solving probability word problems and relate the spontaneous generation of diagrams and other external inscriptions to these stages and their implied subgoals. For example, to solve mathematics word problems, the problem solver must first construct an internal representation of the problem and build a mental model of the problem situation (Casey, 1978, cited in Clements, 1980; Kintsch & Greeno, 1985; Mayer, 1992). Visualization may play a role in this phase of problem solving, and external inscriptions may be used to aid in text comprehension as well as in later, more purely mathematical steps (Corter & Zahner, 2007).

Complex calculations are often made by writing down the numbers involved and following an algorithm for the specific type of calculation. Mathematical and arithmetic inscriptions have been argued to have visual-spatial aspects as well as symbolic content (Kirshner & Awtry, 2004; Landy & Goldstone, 2007; Presmeg, 1986). In the domain of statistics and data analysis, various types of tallies, data organization devices, and visual displays of data may be used (e.g., Tukey, 1977). In the area of probability, certain types of schematic diagrams are conventionally used to represent important concepts (e.g., Venn diagrams for compound events, outcome trees for sequential experiments), and students do exhibit spontaneous use of these standard diagrams in solving problems (e.g., Corter & Zahner, 2007; Russell, 2000; Zahner & Corter, 2002). Use of schematic diagrams, particularly the ones conventionally used in probability instruction such as Venn diagrams, may occur mainly at the latter stages as the problem is cast in mathematical terms and solved.

In the present investigation, we classify the external visual representations made by probability problem solvers using a scheme developed in prior studies (Corter & Zahner, 2007; Zahner & Corter, 2002). This scheme includes examples of schematic diagrams, pictures (iconic) representations, and certain forms of spatial organization and tabulation of problem information. The distinction between schematic and iconic or "pictorial" visual representations is an important one. Schematic representations are those that depict relationships described in the problem, while iconic (pictorial) representations are those that depict the physical appearance of the elements described in the problem. Hegarty and Kozhevnikov (1999) found that the use of schematic representations led to a higher rate of success in a mathematical problem-solving task, whereas use of pictorial representations led to a lower rate of success.

To summarize, external inscriptions may be used for a variety of purposes, including summarizing problem information, recording and reasoning about situation/story elements, offloading memory storage, coordinating the results of intermediate calculations, representing numerical or functional relationships via graphs, and making abstract relationships concrete (cf. Tversky, 2001).

Process Models of Mathematics Problem Solving

In the broader literature on mathematical problem solving, several different cognitive theories or frameworks have been proposed to understand the process of mathematical problem solving. For example, Mayer's (1992) model of mathematical problem solving specifies five different types of knowledge that a problem solver needs in order to solve a mathematics word problem. These types include (1) *linguistic knowledge*, which is the student's knowledge of language, i.e., word recognition and comprehension; (2) *semantic knowledge*, a student's general knowledge of facts about the world (including knowledge about mathematics); (3) *schematic knowledge*, i.e., a student's knowledge of the problem topic and the ability to categorize (either correctly or incorrectly) the problem into a particular problem type; (4) *strategic knowledge*, which is a student's knowledge of how to use the various types of available knowledge in generating, planning, and monitoring the solution of problems, such as setting sub-goals; and (5) *procedural knowledge*, or the knowledge of how to perform a sequence of mathematical operations.

Kintsch and Greeno's (1985) model of how arithmetic and algebraic word problems are solved involves both text processing knowledge and semantic knowledge of mathematics. The main components of their processing model are a set of three types of knowledge structures and a set of strategies. The required knowledge structures are a set of propositional frames, used in translating sentences into propositions, plus schemata that represent properties and relations of sets in general form such as counting and arithmetic operations. Specific procedural knowledge in arithmetic is also assumed to be necessary, for example, knowledge of basic mathematical operations such as addition and subtraction of numbers.

Reusser (1996) proposed a stagewise processing model of mathematics problem solving that includes five consecutive stages: (1) constructing a propositional representation of the problem, (2) creating a situational model, (3) transforming the situation model into a formal mathematical representations, (4) applying the operations to calculate the solution, and (5) interpreting the solution in a meaningful way.

Some elements of these recent models have been anticipated in earlier work. For example, Casey (1978; cited in Clements, 1980) proposed a stepwise model for the solution of mathematics word problems. His model consists of the following steps or stages: (1) question reading, (2) question comprehension, (3) strategy selection, (4) skills selection, and (5) skills manipulation. In this model, the problem solver can "cycle back" to a previous stage to correct errors or try another solution path. Our proposed model of probability problem solving (PPS) is quite similar to Reusser's (1996) and Casey's (1978) models.

However, probability word problems may present certain unique challenges to the would-be problem solver due to the difficulties people experience in probabilistic reasoning and the abstract nature of the material (e.g., Konold, 1989; Mosteller, 1980). Perhaps because the concept of probability seems so abstract to some students, many statistics textbooks emphasize visual representations in their presentations of probability (Russell, 2000). Thus, there may be a special role in this domain for visualization and the use of external graphical representations.

In prior studies (e.g., Corter & Zahner, 2007; O'Connell & Corter, 1993) we have provided evidence that students move through a sequence of problem solving activities as they solve probability word problems. Roughly, students' initial efforts, musings, and inscriptions showed that they are making efforts to understand the problem text and then build a mental model of the problem. Problem solvers then attempt to cast the problem in mathematical terms and possibly relate the current problem to familiar mathematical formulas and/or previously encountered problems. After that, they proceed to develop a plan for solving the problem. Finally, they execute the chosen strategy. A final sub-step that sometimes does and sometimes does not occur is to check the solution for plausibility. These observations have led us to adopt a process model of probability problem solving that includes the following broad stages:

1. Text Comprehension
2. Mathematical Problem Representation
3. Strategy Formulation and Selection
4. Execution of the Strategy

In the present study we analyze problem solvers' written and verbal think-aloud protocols in terms of these problem-solving stages. One of our goals is to investigate how self-generated external visual representations are used by probability problem solvers. Specifically, we examine how frequently and how appropriately these external visual representations are used and whether they facilitate solution success. The main goal, however, is to better understand why use of external visuals may be helpful in problem solving by relating use of specific types of external visual devices to the specific stages of probability problem solving.

Summary of Research Goals

Mathematics problem solvers use a variety of external inscriptions to help them in their work. Previous research in our laboratory (Corter & Zahner, 2007; Russell, 2000) has shown that students solving probability problems use both pictorial and schematic external visual representations in addition to formulas and calculations. Furthermore, students often use spatially organized lists and tallies (as detailed below in Methods). The present study investigates the kinds of external visual representations that are used in probability problem solving (PPS) as well as how and when they are used. One specific issue investigated is whether use of particular types of visual representations is associated with certain types of probability problems (cf. Corter & Zahner, 2007). Another is whether problem complexity affects the use of external visual representations. Two functions that external visual representations can serve are offloading memory storage and helping to organize problem-solving strategies. If this is true, we might expect to see external representations used more often for atypical or complex problems. To these ends, we manipulate the specific probability subtopic and typicality/complexity of the problems and observe how these manipulations affect use of external visual representations and solution success.

We also gather detailed process information, using spoken and written transcripts of problem-solving activity, and use these data to relate use of specific visual representations to specific problem solving stages. In the discussion section, we also attempt to address the question of *why* external visual representations might be useful in PPS.

METHOD

Participants

Thirty-four adult students were recruited from three sections of an introductory graduate statistics course at Teachers College, Columbia University during the spring semester of 2004 to participate in the study. The students in this course are nearly all in the social sciences with applications in education or other education-related programs. Students have diverse backgrounds, ranging from people who were mathematics majors as undergraduates to people who have avoided taking any math courses since the ninth grade. The mean age of course registrants is 28, approximately two-thirds are female, and the mean number of undergraduate math courses taken is 1.9. Participants were volunteers; they received a payment of ten dollars. All three sections of the course used the same textbook (Mendenhall, Beaver, & Beaver, 2003) and the lectures for all three course sections were based on the same curriculum. Participants were informed that they were going to participate in a study of the methods used to solve probability problems.

Materials

A set of 18 probability problems was developed for this study, using six different probability topics: *Joint Events, Conditional Probability, Independent Events, Combinations, Fundamental Principle of Combinatorics,* and *Permutations* and three different variants for each problem topic. The first three topics, *Joint Events, Conditional Probability,* and *Independent Events,* all involve compound or joint events. The last three topics, *Combinations, Fundamental Principle,* and *Permutations,* can be classified as combinatorics. Each topic was represented by a single problem, instantiated in three variants: a “typical” variant, an “atypical” variant, and a “complex” variant. The typical variant was a problem that could be solved using a straightforward application of a standard probability formula explicitly presented in the course curriculum (e.g., the formula defining conditional probability). The atypical variant was a problem that was not isomorphic to any standard problem presented in the lectures or textbook or that could not be solved using a straightforward application of a standard probability formula. For example, an atypical problem in conditional probability might provide a conditional probability and the relevant base rate and ask for the probability of the intersection event. The complex variant could be solved using standard probability formulas and familiar computations; however, it was more complex due to either more extensive calculations or because it required application of several formulas and coordination of multiple subgoals. These manipulations of problem typicality and complexity can be clarified by referring to Appendix A, which presents the six problem topics and their variants, and to Appendix B, which presents possible solution schemas for each problem variant.

Procedure

Participants solved probability word problems working in a paper-and-pencil format, while simultaneously thinking aloud. As part of this interviewing method, a script was developed for the interviewer to prompt the participants when they reached an impasse or if they lapsed into silence while solving the problem (Table 1). Thus, the methodology collects both written and verbal data and uses a structured or clinical interview methodology to elicit more detailed information (Ginsburg, 1997). A single interviewer worked with all of the participants.

Each participant was given a packet that consisted of six probability problems, each presented on a separate page. An incomplete blocked design was used for this study so that each participant only saw one instantiation of each problem topic: two typical variants, two atypical variants, and two complex variants. The assignment of specific variants to topics was counterbalanced, resulting in three different sets of problems (test forms) given to three different subsets of participants. Each test form had six problems, one for each problem topic. The first test form had typical variants of the first and fourth problems, atypical variants of the second and fifth problems, and complex variants of the third and sixth problems. The second test form had atypical first and fourth problems, complex second and fifth problems, and typical third and

TABLE 1
Interviewing Script

Verbal Protocol Issue	*Script*
Can't get started	A. "In general, what would be a good first step in solving this kind of problem?"
	B. "How would you apply it in this case?"
Pauses	A. Wait
	B. "What are you thinking?"
	C. "How did you figure this out?"
	D. Major stuck: "Let's back up and look at this question again. How else could you solve this?"
Lack of detail	A. "Can you explain how you arrived at this?"
	B. "Can you explain what solution method you are using?"
Upon completion	A. Not sufficient detail: "Can you explain all the steps you used to arrive at this answer?"
	B. Sufficient detail: "If you feel you are finished, you may move onto the next problem."

sixth problems. The third test form had a complex first and fourth problems, typical second and fifth problems, and typical third and sixth problems.

The following instructions were given. “Here are six probability problems we would like you to solve. We are more interested in the process of your problem solving rather than the correct answer. So please focus on what you are thinking while you are solving these problems and either write down or say aloud what you are thinking. Try to describe as much of the process you are going through while solving these problems as possible. Please read each question aloud to start.” The interviewer was present during the entire problem solving process and stepped in with verbal prompts if necessary, as specified in the interviewing script (Table 1).

A digital videotape recorder was used to record participants’ verbal and written behaviors. The video portion of the tape captured their written work, including the sequence of problem-solving steps. The paper copy of their written work was also retained for coding. The audio portion of the digital videotape recorded their verbal accounts of their problem-solving process along with the interviewer’s comments and questions.

Coding of the Written Protocols

The coding scheme developed for the participants’ written work was adapted from previous research in our laboratory (Corter & Zahner, 2007; Russell, 2000; Zahner, 2005). Two aspects of the written problem solutions were coded. The first aspect coded whether the participant gave the correct answer. This was simply coded as 0 for incorrect and 1 for correct. The second aspect coded for the type of external visual representation used (if any) by the participant. The identified types were *spatial reorganization of the given information, outcome listings, contingency tables, Venn diagrams, trees, novel graphical representations,* and *pictures.* These types are defined below.

Types of External Visual Devices

An external visual representation was coded as a *picture* if it attempted to represent the real-world situation conveyed in the problem in a non-symbolic, pictorial way. For example, in a problem about selecting compact discs (CDs), any pictorial representation of a CD would count as a *picture* (see Figure 1 for an example). A visual device was coded as an *outcome listing* if it gave a list of outcomes in some relevant outcome space, for example: {HH, HT, TH, TT} as the outcome space for the experiment of flipping a coin twice. A visual representation was coded as a *tree* diagram if the participant attempted to organize the information from the problem into an outcome tree (e.g., Figure 2). A visual representation was coded as a *contingency table* if the participant presented the information from the problem as probabilities or frequencies in a two-way table. Use of a *Venn diagram* was coded if the participant used a Venn diagram to represent set relationships (e.g., Figure 3).

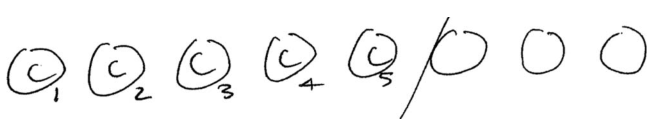

FIGURE 1 A participant’s written work for the typical version of the *Combinations* problem, illustrating use of a *picture*.

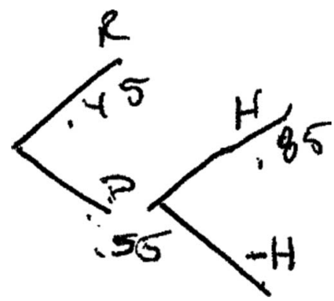

FIGURE 2 A participant's written work for the typical version of the *Conditional Probability* problem, illustrating use of an outcome tree.

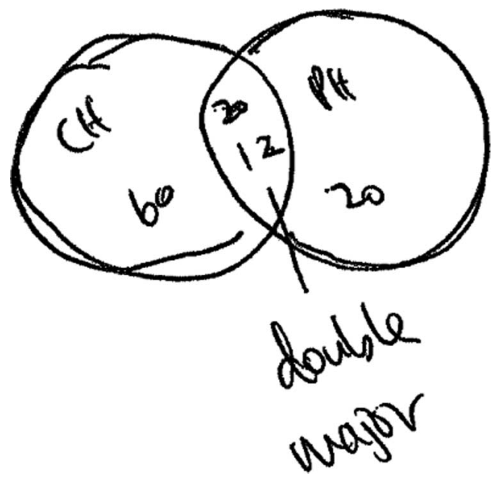

FIGURE 3 A participant's written work for the typical version of the *Joint Events* problem, illustrating the use of a Venn diagram.

These five types of external visual representations had been explicitly introduced to the students in their probability lectures. Two additional coding categories were defined to cover cases not handled by the above types. These two categories were identified in a previous study (Corter & Zahner, 2007) of the use of external visual representations in probability problem solving. The first is a code indicating any attempt to invent and use a "*novel" schematic representation*, defined as a representation not taught in the introductory class the participants were taking nor used frequently in standard probability texts (Russell, 2000). Note that these "novel" external visual representations include conventional types of graphs and diagrams possibly known to the participants through other courses and experiences; however, they are here designated as novel because they were not taught in the probability curriculum of the participants' course. For example, several participants used a bipartite graph for the *Independent Events* problems, consisting of a list of three factories connected by lines or arrows to probabilities (Figure 4); also, several participants used a graph in which the cardinality of a set of paths is represented by number labels

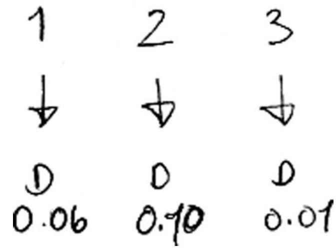

FIGURE 4 A participant's written work for typical version of the *Independent Events* Problem, illustrating use of a novel schematic representation.

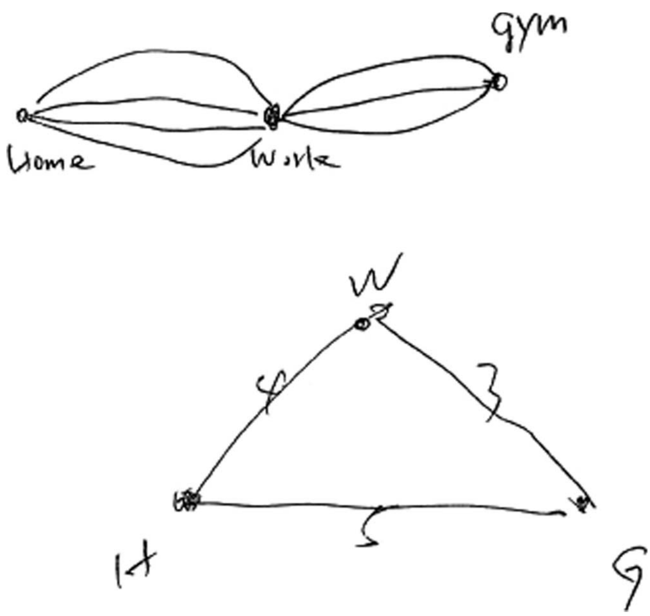

FIGURE 5 A participant's written work for the complex variant of the *Fundamental Principle* problem, illustrating use of another type of novel schematic representation and a picture. Note that the diagrams here were initially coded as both a *picture* and a *novel schematic* representation, but after discussion between the coders, the consensus was that these are not simply pictures due to the schematic nature (lines for roads) of the representation.

(Figure 5) for the *Fundamental Principle* problems. These types of graphical representations use schematic elements (lines or arcs) to represent abstract relations in the problem and are thus distinguished from a simple outcome listing or a picture. One exception to this neat separation of pictures and schematic representations arises with the specific "Fundamental Principle" problem that was used that involves different routes among physical locations. A simple abstract picture of this problem (see the upper half of Figure 5) can also be considered to be a graph, consequently, some such indeterminate instances were initially coded as both a picture and a novel schematic representation.

The second additional coding category was defined to include any *spatial reorganization of the given information*. Use of a spatial organization scheme for information is neither a formal graphical representation nor a purely pictorial representation. However, we have included this coding category because we have observed frequent use of spatially aligned rewriting of information to aid in problem solving (cf. Kirshner & Awtry, 2004; Landy & Goldstone, 2007; Presmeg, 1986). For example, in the present study many participants were observed to vertically align related pieces of given information (e.g., conditional probabilities), creating a type of informal table (see Figure 6). This reorganization may make it easier for novice problem solvers to check for needed or missing information, to break down problem solution into subparts, or to make visual associations to relevant formulas.

In order to assess the reliability of this coding scheme for external representations, a second rater was trained and coded all participants' written solutions. Initial reliability of the coding of type of external visual representation, as measured by Cohen's kappa, was equal to .98. The few discrepancies between the two raters were all related to the *novel schematic representation* code for the *Fundamental Principles* problem. One coder initially coded figures similar to the ones in Figure 5 as a *picture*, whereas the other coder initially coded those types of figures as a *novel schematic representation.* The two raters discussed these discrepancies and reached consensus by coding such figures as a *novel schematic representation*, since these diagrams can be considered to be graphs whose arcs representing possible travel between pairs of nodes (representing destinations).

Coding of Verbal Protocols and Interviews

The coding scheme for the participants' think-aloud verbal protocols was developed in order to analyze problem solvers' utterances in terms of *problem-solving stages* and to investigate how and when problem solvers are using external visual representations. For this coding scheme, we transcribed and examined the audio portion of the videotapes capturing the participants' think-aloud verbal protocols. First, the verbal protocols were parsed into utterances. Each utterance was then coded to indicate the problem-solving stage in which the participant was engaged, text comprehension, math problem representation, strategy formulation, or execution of a solution.

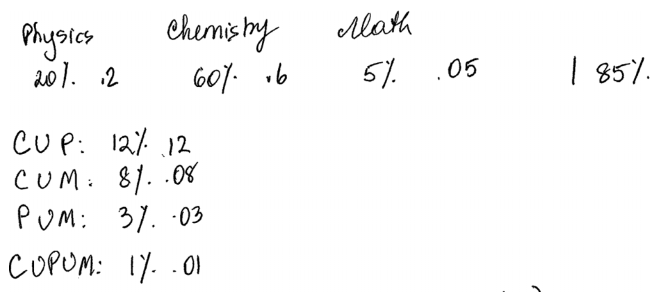

FIGURE 6 A participant's written work for the atypical version of the *Joint Events* problem, illustrating use of spatial *reorganization of given information*.

Because the utterances differed in length, we counted the number of words that were used within each utterance in order to measure the approximate amount of time spent in each utterance and by extension in each processing stage. At the level of individual utterances, the processing stages were coded as mutually exclusive.

An utterance was coded as text comprehension if the participant was reading the words of the probability problem and attempting to use the verbal description to build an understanding of the real world context. For example, if a participant was reading the problem text and underlining key words, that utterance would have been coded as text comprehension. An utterance was coded as math problem representation if the participant was attempting to relate the real-world situation described in the problem to probability or other mathematical concepts. For example, if the participant rewrote the statement "given that Democrat voted for H" as "P(D|H)" (probability of D given H), this would have been coded as math problem representation. An utterance was coded as strategy formulation if the participant was considering or developing a strategy to solve the probability problem. For example, if the participant said, "I'm going to do the formula of this plus this plus this minus this [formula for P(P or C)]" (Subject #5), the unit was coded as strategy formulation and selection. Finally, an utterance was coded as execution of a solution if the participant was actively solving the problem, for example, instantiating a formula or doing calculations.

The video tracks of the videotapes were used to temporally match the participants' production of external devices with their verbal statements. In order to assess reliability of the coding of the audio and video protocols, a second rater was trained and coded all participants' audio and video protocols. Initial reliability was approximately 87% (Cohen's K = .872). After additional training, the inter-rater reliability improved to 94% (Cohen's K = .943). The two raters then discussed the remaining discrepancies and the resulting consensus coding was used in all analyses reported.

RESULTS

An initial step in the analysis was to code and record how often each type of external visual representations was used by the participants. As shown in Figure 7, for these six topics participants

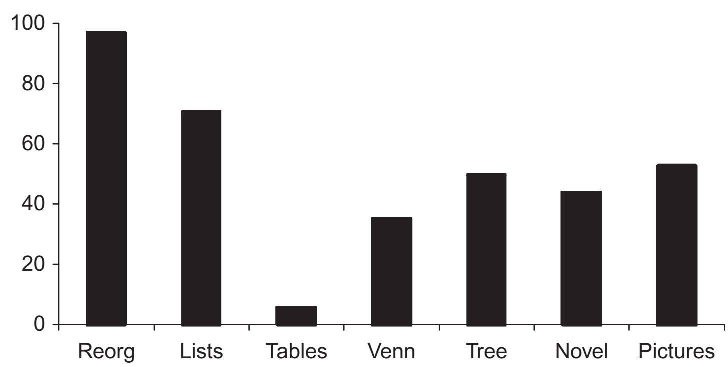

FIGURE 7 Percentage of participants (N = 34) using each type of external visual representation at least once.

most often used *spatial reorganization of the given information* (by 97% of the participants), followed by use of *outcome listings* (by 71% of the participants), *pictures* (53% of the participants), *trees* (50%), *novel schematic representations* (44%), *Venn diagrams* (35%), and finally *contingency tables* (6%). These results show that problem solvers do spontaneously create external visual representations when solving probability problems.

Use of External Visual Representations for Different Problem Topics

If probability problem solvers use external visual representations to facilitate problem solving, we might expect specific types of diagrams to be used for specific problem types or topics (cf. Novick & Hurley, 2001). Thus, we coded and analyzed the relationship between topic of the probability problem (*Joint Events, Conditional Probability, Independent Events, Permutations, Fundamental Principle,* and *Combinations*) and the type of representation (if any) that participants chose to use. In this analysis no distinction was made between the different variants of the problems (typical, atypical, and complex). Table 2 summarizes how often each type of external representation was used for each problem topic.

If a given type of representation is used differentially often across problem topics, then this should show up as an association between use of each specific external visual representation and problem topic. For each type of external visual representation (i.e., for each row of Table 2), we applied Cochran's test for the homogeneity of proportions in a k by 2 table (Cochran, 1950) to test for the use or no use of a representation across problem topics. Cochran's Q is designed to test for differences in a dichotomous outcome variable across k conditions in a matched-samples design. The tests showed significant differences in the use of each external visual representation across problem topics, with the sole exception of *contingency table*. However, contingency tables were used for only two problem instances (Table 2). *Venn diagrams* were used only for the Compound Events (P1 – P3) problems and not at all for the Combinatorics problems (P4 – P6). In contrast to the *Venn diagram*, the use of an *outcome listing* was used almost exclusively for

TABLE 2
Frequency of Use of Each Type of Representation by Problem (Averaged Across Versions). Each Row Also Shows Results for Cochran's Test for Homogeneity of Proportions for Use of That Representation Across the Six Problems (Each Cell Frequency is Based on N = 34 Problem Solutions; Cells with the Complementary Frequencies of Problems NOT Using the Representation Are Not Shown)

	Compound Events			*Combinatorics*				
Representation	*P1 Joint Events*	*P2 Cond Prob*	*P3 Ind Events*	*P4 Perm*	*P5 Fund Princ*	*P6 Comb*	*Cochran's Q (df = 5)*	*p-value*
Reorganize	28	25	23	8	8	21	45.28*	.000
Listings	0	0	2	17	7	12	44.42*	.000
Tables	0	1	0	0	0	1	4.00	.549
Venn Diagrams	12	3	2	0	0	0	43.53*	.000
Trees	0	5	6	2	10	5	15.61*	.008
Novel Schematic	2	0	7	1	12	4	15.00*	.010
Pictures	0	0	0	0	10	7	42.85*	.000

*p < .05.

the Combinatorics problems and by two participants for Problem 4. Other representations (for example, *trees*) are used across most of the problem topics, but in differing proportions. These results indicate that specific representations are used more or less often for the different problem topics.

Note that some types of representations are used for many different types of problems. For example, we believe that *reorganization of the given* information is used across all problem topics because it is a very general strategy—both in that it does not correspond to any one type of problem "schema" and because it is a strategy that can be used to reduce cognitive load and/or to help problem solvers extract the necessary mathematical information from the given word problem. Reorganization may aid in the abstraction of a problem schema from the text of a word problem in part by selecting out the critical problem information from the mass of superficial story detail. It is perhaps not a coincidence that the two problems where *reorganization* was used least often were the problems (P4: Permutations & P5: Fundamental Principle) that had the least amount of text.

Trees were also used across nearly all of the problem types, with the exception of problems involving joint events. Outcome trees can in fact be used for multiple types of problems (e.g., for problems involving conditional probabilities or for sequential events), and appeared frequently in the curriculum of the course. Because trees can be used for many different types of problems involving compound events, it is interesting that they are never used for the joint events problems. We suspect that this is because trees are conventionally used for problems that involve temporal ordering (e.g., two sequential flips of a coin). All of the problems involved in the present study, with the exception of *joint events* (P1) involve separable events that could (although did not have to) be viewed as being ordered either temporally (P3-P6) or causally (P2). Because the joint events problems did not involve (or admit of) temporal ordering of the constituent events, they may not have cued retrieval of the *trees* schema. We return to these speculations in the Discussion.

Problem Complexity and Representation Use

One research question raised earlier is whether external representations are used more often for atypical or complex variants of problems. To test this idea, an analysis was conducted to check for the homogeneity of proportions (Cochran, 1950), for the use or no use of a representation by problem variant (*typical, atypical,* and *complex)* for each type of representation. Table 3 shows the frequency of use of each type of external visual representation by type of problem variant. A test of the homogeneity of proportions (use or no use of a representation across the three types of problem variant) revealed that the use of *outcome listings* was significantly different for the 3 problem variants, Cochran's $Q(2) = 12.50$, $p = .002$. It appears that *outcome listings* were used less often for atypical problems. Venn diagrams tended to be used more often for the complex problems, Cochran's $Q(2) = 6.22$, $p = .045$. This pattern suggests that Venn diagrams may be more often used to coordinate complex information, such as problems involving compound events. No other significant associations were found between problem variant and use of a given external visual representation.

Solution Success and Use of External Visual Representations

If external visual devices are indeed useful to problem solvers, then we should expect a positive association between solution success and the specific external representation used (if any). As

TABLE 3
Frequency of Use of a Particular External Visual Representation by Variant, With Cochran's Test for Homogeneity of Proportions for Use of That Representation Across Three Problem Variants (For Each Cell, N = 34 Participants; the Cells with the Complementary Frequencies Are Not Shown)

Representation	*Variant*			*Cochran's Q (df = 2)*	*p-value*
	Typical	*Atypical*	*Complex*		
Non-diagrammatic					
Reorganization	28	30	31	2.33	.311
Outcome Listings	17	7	17	12.50*	.002
Schematic Diagrams					
Contingency Tables	0	0	1	2.00	.368
Venn Diagrams	6	4	10	6.22*	.045
Trees	12	8	12	3.20	.202
Novel Schematics	6	9	9	1.64	.441
Iconic					
Pictures	3	8	3	4.55	.103

*p < .05.

TABLE 4
Conditional Probability of a Correct Solution Given the Use of a Particular Representation, Separately by Problem Topic (With Number of Relevant Observations Shown in Parentheses)

Representation	*Compound Events*			*Combinatorics*		
	P1 Joint Events	*P2 Cond Prob*	*P3 Ind Events*	*P4 Perm*	*P5 Fund Princ*	*P6 Comb*
Non-diagrammatic						
Reorganize	.179 (28)	.480 (25)	.522 (23)	.625 (8)	.375 (8)	.286 (21)
Listings	–	–	–	.529 (17)	.571 (7)	.167 (12)
Schematic Diagrams						
Tables	–	–	–	–	–	–
Venn	.333* (12)	–	–	–	–	–
Trees	–	.800* (5)	.667 (6)	–	.700 (10)	.600*(5)
Novel	–	–	.571 (7)	–	.917* (12)	.500 (4)
Iconic						
Pictures	–	–	–	–	.700 (10)	.143 (7)
(Baseline)						
Mean P(correct):	**.176**	**.529**	**.559**	**.559**	**.676**	**.294**

* = significantly higher (p < .05) than mean performance for that problem topic, by a binomial test.
– Dashed lines indicate a cell with fewer than four uses of that representation (i.e., n ≤ 3).

shown in Table 4, the use of certain external visual representations was associated with significantly higher rates of solution correctness for certain problem topics (compared to baseline for that problem topic). For the *Joint Events* problems, participants who used a *Venn diagram* had a significantly higher rate of solution success than the average rate for these problems. For the

Conditional Probability and *Combinations* problems, participants using a *tree* had a significantly higher rate of solution success than the mean rate for these problems. For the *Fundamental Principle* problems, participants who used a *novel schematic representation* had a significantly higher rate of solution success than the baseline rate.

Table 4 documents a few cases in which use of an external visual representation seems to be associated with a lower rate of solution success. However, none of these "reverse" correlations was significant. For example, for the Combinations problems, use of pictorial or non-diagrammatic external visual representations was associated with an apparently lower rate of solution success. This outcome, although not significant, is consistent with the results of studies by van Garderen and Montague (2003) and Hegarty and Kozhevnikov (1999), who found that the use of schematic diagrams was correlated with solution success whereas use of non-schematic or pictorial diagrams was negatively correlated with solution success. In sum, the results summarized in Table 4 suggest that choosing an appropriate external visual representation is important in problem solving.

Stages of Probability Problem Solving

One important goal of the present project was to investigate how external visual representations are used in the solution process. As a preliminary step, we derived an approximate measure of the time spent in each processing stage, by counting the number of words in the utterances associated with each stage. The results are shown in Figure 8. It can be seen that participants spent only about 5% of their time (as measured by number of utterances) in reading and understanding the problem text. In contrast, they spent 56% of their time in math problem representation, and 19% of their time in strategy formulation, with only 20% in execution. Of course, the number of words is only a rough proxy for processing time; thus these results must be considered to be merely approximate.

These stages of probability problem solving are not always followed in a strict linear order. To explore this idea, we tabulated all transitions between two different stages as shown by the coding of sequential utterances (Figure 9). The most typical "path" through the stages starts with text comprehension, followed by math problem representation and strategy formulation (sometimes with "shuttling" back and forth between these two stages). Finally, the problem solver moves to execution of a strategy, either from math problem representation or strategy formulation. The

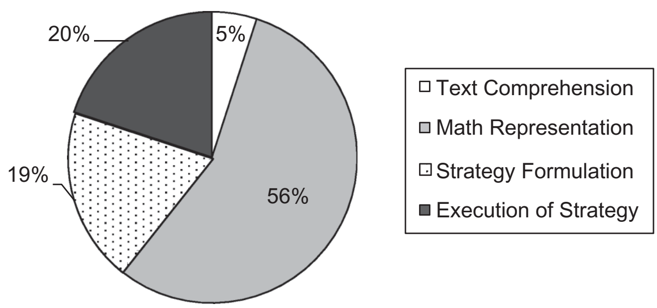

FIGURE 8 The mean percentage of time spent in each problem-solving stage (N = 208 problem solutions).

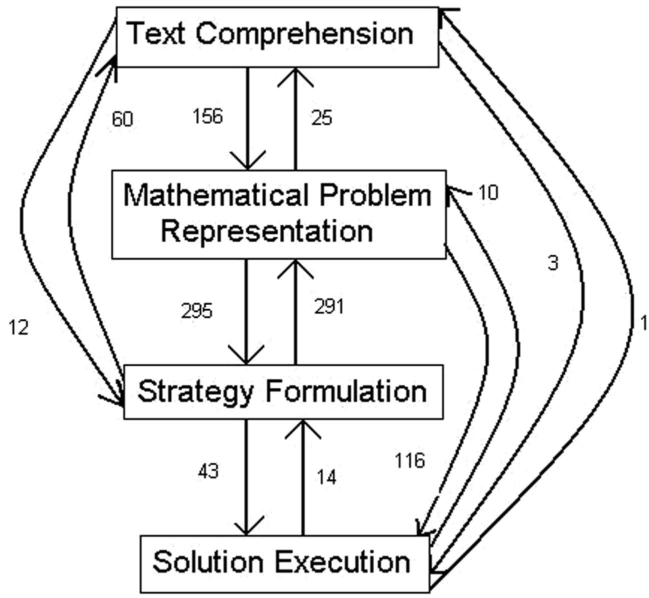

FIGURE 9 The four problem-solving stages, with observed transition frequencies shown (N = 208 problem solutions; 1690 total transitions).

observed frequent shuttling between the math problem representation stage and the strategy formulation stage indicates that the problem-solving process is not comprised of a fixed number of linear sequential stages, but that solving problems can involve iterated solution attempts (cf. Casey, 1978).

Use of Different Types of External Visual Representations in Different Stages of PPS

An important goal of the present study was to investigate whether different types of external visual representations are used to support different stages of the problem-solving process model. If this is so, then the probability of using a particular representation during a given utterance should differ depending on the problem-solving stage. Table 5 shows how the frequency of first use for each type of external visual representation is distributed across stages. Note that nearly all of the external representations have their maximal frequency of creation and use in Stage II: Math Problem Representation. The exceptions are use of *lists* and *trees*, which show maximal initial use in Stage III: Strategy Selection. The second-most common stage for creation and use of each external representation also tends to be these two stages, except for *tables*, which are sometimes created and used in Stage IV: Execution, and *pictures*, which are often created in Stage I, *Text Understanding*.

To check if these differences in use of external visuals across stages are significant, a log-linear analysis was conducted on the distribution of n = 2133 process units in the transcripts, classified by the four problem-solving stages, the six problem topics, and the seven types of external representation. The stage by representation interaction was significant, indicating that certain types

TABLE 5
Frequency of Use of Each External Visual Representation by Processing Stage

	I: Text	*II: Math*	*III: Strategy*	*IV: Execution*
Non-diagrammatic				
Reorganize	26	207	57	1
Listings	0	8	52	3
Schematic Diagrams				
Tables	0	7	2	4
Venn	3	29	4	0
Trees	3	14	53	1
Novel	2	23	15	0
Iconic				
Pictures	7	18	1	0
Total	41	306	184	9

of external visual representations are used more often during particular problem solving stages, $\chi^2(18) = 219.07$, $p < .05$. This result demonstrates that different types of external representations have their primary uses in different processing stages.

There was also a significant association between problem-solving stage and problem topic, $\chi^2(15) = 26.18$, $p < .05$, indicating that participants spend differing lengths of time in a particular problem-solving stage depending on the problem topic. Finally, there was a significant association between problem topic and type of external visual representation, $\chi^2(30) = 432.12$, $p < .05$, consistent with earlier analyses (see Table 2).

DISCUSSION

Our results show, first, that students spontaneously create external inscriptions while solving probability word problems. Presumably this is because these inscriptions are useful in solving the problems, since participants were requested merely to “show their work” and were not explicitly instructed to produce diagrams or other visual devices. Our analyses relating solution success to spontaneous use of the external spatial representations also demonstrate the utility of these representations for probability problem solvers.

The specific types of external visual representations used in probability problem solving that we identified are (in decreasing order of frequency of use): *spatial reorganization of the given information, outcome listings, pictures, trees, novel schematic representations, Venn diagrams,* and *contingency tables.* These seven types of external visual representations may be classified as being of three general types: *pictures* (iconic), schematic diagrams (*contingency tables, trees, novel schematic representations,* and *Venn diagrams*) and spatially organized but non-diagrammatic forms of tallying (*spatial reorganization of the given information and outcome listings*).

Students solving mathematics problems create other types of external inscriptions as well, most notably equations and computations. We did not systematically study such inscriptions related to the formal language of mathematics because our primary interest was in inscriptions that use *spatial information* to code aspects of the problem. Visually based external inscriptions presumably reflect visual reasoning processes, long believed to be important in mathematical thought.

Additionally, the present results lend support to the idea that the process of probability problem solving can be resolved into stages that usually but not always occur sequentially, with repeated shuttling between some of the stages. The stages, in typical order of occurrence, are: text comprehension, mathematical problem representation, strategy formulation and selection, and execution of the strategy. Shuttling back and forth between mathematical problem representation and strategy formulation and selection is frequently observed. These results (although primarily descriptive in nature) are consistent with previous models of mathematics problem solving (e.g., Anderson, 1996; Casey, 1978; Kintsch & Greeno, 1985; Mayer, 1992; Reusser, 1996; Schoenfeld, 1994). Furthermore, our micro-level analyses of problem-solving behavior results have allowed us to show associations of particular types of external visual representations with particular stages of probability problem solving.

Limitations

Our conclusions may warrant some caution, because there are potential limitations to the methodology we have employed. First, the population studied here was composed of students drawn from an introductory statistics course at a graduate school of education; they were adults, with diverse mathematical backgrounds. Furthermore, participants were volunteers and were compensated with a modest payment, so it is not guaranteed that they are typical of all students in this course, nor of other populations who are learning probability. Another possible objection that might be raised to our method is that students might solve probability problems differently in the laboratory and in the classroom due to the requirement that they think aloud while solving problems or due to other demand characteristics of the lab setting. However, several decades of research has supported the validity of data from think-aloud protocols in investigating mathematics problem solving, even with elementary school students (e.g., Ericsson & Simon, 1993; Robinson, 2001). Thus, we do not view these potential threats to generalizability as fatal, but further research will be needed to ensure that our results indeed generalize to other populations and settings.

One specific aspect of our results that may not generalize is the overall proportions of various types of external visual representations that we observed. Not only do the frequencies of use of these representations depend on the particular problems chosen (as shown by our results, e.g., Table 2), but these frequencies may also be strongly affected by the particular curriculum used in the course in which the students were enrolled. In this curriculum, outcome listings, outcome trees, Venn diagrams, and contingency tables were used frequently. Thus, the results we have obtained about the frequency of use of various external visual devices may be curriculum-dependent. However, it does not seem that participants were simply reflecting course or textbook practices, because (for example) contingency tables, although used frequently in the text and course, were almost never used by participants in this study. This gives us confidence that we are seeing “signal” (i.e., students’ true predilections regarding use of specific external inscriptions for specific problems) in the “noise” (i.e., bias due to curriculum or to the special conditions of the laboratory study).

Why Are External Visual Representations Useful?

The present results provide evidence suggesting that external visual representations are useful in probability problem solving, even when these representations are spontaneously created by the problem solver. In particular, when participants use certain types of external representations with particular problems (presumably problem-appropriate representations), higher rates of solution success are observed compared with using no diagram. However, use of some representations are not positively associated with solution success for some type of problems, which implies that inappropriately chosen representations do not lead to higher rates of solution success. This pattern of results lends support to the idea that choosing an appropriate representation should be viewed as an important subskill in problem solving (Novick & Hmelo, 1994). Of course, both use of diagrams and solution success are dependent variables in this study, thus firm causal conclusions regarding the observed associations between them are not possible; it may be that correct problem understanding by the student results in better choice of a visual representation rather than vice-versa. However, if this latter interpretation is accepted, then the question arises of why a student should choose to produce a diagram at all in solving the problem. Thus, we believe that the most plausible interpretation of our results is that selection and use of an appropriate diagram is an aid to problem solving, and that students spontaneously produce external visual representations in problem solving because it is helpful to them.

Is there a general answer as to why diagrams and other visualizations are useful in mathematics problem solving? Visual reasoning is believed to be associated with discovery in mathematics problem solving (Hadamard, 1945; Polya, 1957). Thus, to the expert mathematician, diagrams may be most useful for exploration of non-routine problems (Pantziara, Gagatsis, & Elia, 2009), and inventing novel diagrammatic representations is part of the creative process. In the present context, the probability problems encountered all had solutions obtainable using formal methods taught in the course, and standard uses of diagrams (e.g., Venn diagrams) were taught for such problems. In this sense, the presented problems were “routine,” although we manipulated problem typicality and complexity to see if these factors would affect diagram use. We did not find many differences in diagram use associated with these factors, however. Of course, to a novice, even standard problems are not yet routine; thus, diagrams may be most useful when the student is challenged by working near their boundary of competence, the “zone of proximal development” (Vygotsky, 1978).

In general, we believe that what makes a diagram useful to a novice problem solver is both appropriateness to the problem and need: that is, the problem at hand must be difficult or complex enough that the diagram is needed and has a chance of having a facilitative effect. External inscriptions, including but not limited to pictures and diagrams, may be useful aids in problem solving for a number of distinct reasons. First, the use of inscriptions helps to organize the given information in the problem and facilitates the building of a mental model of the problem text. We see evidence of this happening when the participants write or draw pictures on the page during the text comprehension stage. Problem complexity may play a role in this and later stages, because complex problem scenarios involve understanding and manipulating large amounts of given information and often involve the coordination of multiple subgoals (cf. Dean, 2006; Tatsuoka, Corter, & Tatsuoka, 2004). Inscriptions are known to be useful for offloading the results of intermediate calculations from memory, especially in complex problems (Schreiber, 2004).

It is possible that the use of external visual representations helps the participants abstract the problem text to just the important elements and helps them build a problem schema and solve the problem. Inscriptions can also be useful during the math problem representation stage, in which participants match the problem text schema to a familiar probability problem schema, such as definitional formula for conditional probability. Exploration of this idea of diagrams as an aid to schema matching seems a fruitful avenue for future research. Some comments and speculations on this idea are presented in the next section.

Schematic and Non-Schematic Visual Representations

As already described, the seven types of external visual representations studied here differ in some important ways. Four of the representations (*Venn diagrams, trees, contingency tables,* and *novel schematic representations*) we classify as schematic diagrams. Two of the visual representations (*spatial reorganization of given information* and *outcome listings*) can be considered forms of tabulation. The final type (*pictures*) refers to iconic representations of problem elements (e.g., pictures of cars or CDs).

The first group (*contingency tables, Venn diagrams, trees,* and *novel schematic representations*) are considered as schematic because (1) the graphs have a structure or syntax, and (2) structural aspects of the graphs symbolically represent meaningful aspects of the problem. We classify two of the types of external representations (*spatial reorganization of the given information* and *outcome lists*) as non-diagrammatic. In these representations spatial location is used simply to organize lists of comparable information (e.g., list of outcomes or corresponding pieces of information relating to multiple subgoals, as in Figure 6) Finally, *pictures* are classified as iconic representations, in which spatial aspects of the inscriptions represent spatial relationships between problem objects, and *resemblance* is key (cf. Goodman, 1976). Our speculations on diagram use as schema matching apply most directly to the schematic diagrams, trees, contingency tables, Venn diagrams, and certain novel representations.

Schema-Matching in Diagram Use

Novick and co-workers (Hurley & Novick, 2010; Novick, 1990; 2002; Novick & Hmelo, 1994; Novick & Hurley, 2001; Novick, Hurley, & Francis, 1999) have proposed specific semantic or structural features that characterize certain types of diagrams. Novick and Hurley (2001) proposed that different types of schematic diagrams have structural aspects or properties that determine their range of applicability. Our results can be interpreted within this viewpoint. The associations we have found between use of the different types of representations and specific problem types suggest that properties of the diagrams and properties of the problem schema are being matched (although not always successfully) by participants. For example, trees seem naturally appropriate for sequential problems such as the results of multiple coin flips, while contingency tables and Venn diagrams are particularly appropriate for representing joint or compound events. Our model of probability problem solving is based on an assumption that applied probability problems (and the cover stories that are built over them) are structured according to implicit schemas. For example, providing a student with the probability of a joint event $P(A \cap B)$ and the base rate $P(B)$ makes it easy and natural to ask for the corresponding conditional probability of A given B. In addition to such problem text and math problem schemas, several of the types of external visual representations that we have examined have complex structure and thus could be characterized as “schematic” as well. In a schematic diagram, such as a Venn diagram, visual aspects of the representation are used to represent abstract features of the probability problem. For example, in an outcome tree representation of a coin-flipping problem the root node represents the first coin flip, and the two branches leading from the root node represent the two possibilities resulting from that flip (Heads or Tails). Thus, finding an appropriate diagrammatic representation for a problem can be seen as a schema matching process.

In our study we found little use of “novel” graphical representations (that is, of graphical representations that are not conventionally used in probability problems; see Russell, 2000). These were restricted mainly to scattered use of bipartite graphs and the use of general network graphs to represent the Roads problem. These novel uses tended to be incorrect. This observation does not mean that invention is bad; rather, it reflects the fact that the types of schematic representations typically taught in probability courses (Venn diagrams, trees, and contingency tables) are used because they are especially appropriate to represent typical textbook problems. As an example, outcome trees are often found in statistics texts (Russell, 2000) and are known to be widely applicable in probability problem solving. Trees are useful for both conditional probability problems as well as those related to sequential events (roll a die, then flip a coin). They can be used to represent applications of the fundamental principle of combinatorics as well. Consider a problem where a person has three tee-shirts and four pairs of shorts and the problem asks for the number of outfits that can be made with this set of clothing. While a bipartite graph could be used to represent this problem, the resulting graph is visually cluttered and does not make salient the number of possible outcomes. The tree diagram for this problem is much more efficient and less cluttered, and a salient aspect of the diagram, the number of leaves of the tree, corresponds to the goal quantity of the problem, the number of outcomes; therefore, the tree is more useful for solving this problem.

In future work we hope to explore implications of the view that just as problems have a particular schema or structure, so do schematic diagrams. When the two schemas match well, the visual representation may be a useful tool that facilitates problem solving, perhaps leading to a higher rate of solution success. Ross and colleagues (Ross, 1984, 1989; Ross & Kennedy, 1990) present evidence that experts are adept at matching a problem’s structure with a solution structure, as compared to novices, who tend to match problems based on surface structures (e.g., drawing five cards out of a deck). If external visual representations are useful in problem solving and they have specific characteristics that match different schemas, then it is logical that certain external visual representations are more appropriate for certain problem types because matching problem elements to the schematic elements of the external visual representation would be easier for some cases than others.

Our findings showing higher solution rates for certain problems given use of certain visual representations suggest that instruction in probability problem solving ought to include instruction in appropriate schematic representations for probability problems. It is possible that structural aspects of these diagrams should be emphasized in order to help students develop better intuitions about when each type of diagram may be useful. It is less clear whether performance in solving probability or other words problems can be facilitated by encouraging problem solvers to draw iconic representations of the problem text or to use spatial reorganization of the given problem information—our results show no facilitative effects for these types of external inscriptions. Of course, when students are allowed to make external inscriptions, they will no doubt as a group employ a variety of approaches and representations, and this is a good thing. As we try to develop methods to teach appropriate use of diagrams for problem solving, we should be careful not to discourage students from taking initiative and actively trying new representations and new approaches.

REFERENCES

Anderson, J. (1996). ACT: A simple theory of complex cognition. *American Psychologist, 51*(4), 355–365.

Campbell, K. J., Collis, K. F., & Watson, J. M. (1995). Visual processing during mathematical problem solving. *Educational Studies in Mathematics, 28*(2), 177–194.

Casey, D. P. (1978). Failing students: A strategy of error analysis. In P. Costello (Ed.), *Aspects of motivation* (pp. 295–306). Melbourne: Melbourne Mathematical Association of Victoria.

Clement, J., Lochhead, J., & Monk, G. S. (1981). Translation difficulties in learning mathematics. *The American Mathematical Monthly, 88*(4), 286–290.

Clements, M. A. (1980). Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics, 11*(1), 1–21.

Cochran, W. G. (1950). The comparison of percentages in matched samples *Biometrika, 37*, 256–266.

Corter, J. E., & Zahner, D. (2007). Use of external visual representations in probability problem solving. *Statistics Education Research Journal, 6*(1), 22–50.

Dean, M. J. (2006). Explaining performance in the Third International Math and Science Study (TIMSS) 195 Advanced Mathematics Test. (Doctoral Dissertation, Columbia University, 2006). *Dissertation Abstracts International, 67*(4), 1305.

de Hevia, M.-D., & Spelke, E. S. (2009). Spontaneous mapping of number and space in adults and young children. *Cognition, 110*(2), 189–207.

Diezmann, C. M. (1995). Evaluating the effectiveness of the strategy 'Draw a diagram' as a cognitive tool for problem solving. In B. Atweh & S. Flavel (Eds.), *Proceedings of the 18th Annual Conference of Mathematics Education Research Group of Australasia* (pp. 223–228). Darwin, Australia: MERGA.

Douville, P., & Pugalee, D. K. (2003). Investigating the relationship between mental imaging and mathematical problem solving. In A. Rogerson (Ed.), *Proceedings of the International Conference of the Mathematics Education into the 21st Century Project* (pp. 62–67). Brno, Czech Republic: Mathematics Education into the 21st Century Project.

Edens, K., & Potter, E. (2008). How students "unpack" the structure of a word problem: Graphic representations and problem solving. *School Science and Mathematics, 108*(5), 184–196.

English, L. D. (1997). *Mathematical reasoning: Analogies, metaphors, and images*. Mahwah, NJ: Lawrence Erlbaum.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: Verbal reports as data*. Cambridge, MA: MIT Press.

Ginsburg, H. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, UK: Cambridge University Press.

Goodman, N. (1976). *Languages of art*. Indianapolis, IN: Hackett.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. New York: Oxford University Press.

Hall, V., Bailey, J., & Tillman, C. (1997). Can student-generated illustrations be worth ten thousand words? *Journal of Educational Psychology, 89*(4), 677–681.

Hannafin, R. D., Burruss, J. D., & Little, C. (2001). Learning with dynamic geometry programs: Perspectives of teachers and learners. *The Journal of Educational Research, 94*(3), 132–144.

Hannafin, R. D., & Scott, B. (1998). Indentifying critical learner traits in a dynamic computer-based geometry program. *The Journal of Educational Research, 92*(1), 3–12.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684–689.

Hollebrands, K. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *Journal of Mathematical Behavior, 22*(1), 55–72.

Hurley, S. M., & Novick, L. R. (2010). Solving problems using matrix, network, and hierarchy diagrams: The consequences of violating construction conventions. *The Quarterly Journal of Experimental Psychology, 63*(2), 275–290.

Kaufmann, G. (1990). Imagery effects on problem solving. In P. J. Hampson, D. E. Marks, & J. T. E. Richardson (Eds.), *Imagery: Current developments* (pp. 169–197). New York: Routledge.

Kintsch, W., & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review, 92*(1), 109–129.

Kirshner, D., & Awtry, T. (2004). Visual salience of algebraic transformations. *Journal for Research in Mathematics Education, 35*(4), 224–257.

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6*(1), 59–98.

Koedinger, K. R., & Anderson, R. (1997). Intelligent tutoring goes to school in the big city. *International Journal of Artificial Intelligence in Education, 8*(1), 30–43.

Landy, D., & Goldstone, R. (2007). Formal notations are diagrams: Evidence from a production task. *Memory & Cognition, 35*(8), 2033–2040.

Latour, B., & Woolgar, S. (1986). *Laboratory life: The social construction of scientific facts*. Princeton, NJ: Princeton University Press.

Lean, G., & Clements, M. A. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics, 12*(3), 267–299.

Lehrer, R., Schauble, L., Carpenter, S., & Penner, D. E. (2000). The inter-related development of inscriptions and conceptual understanding. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 325–360). Mahwah, NJ: Lawrence Erlbaum.

Mayer, R. (1989). Systemic thinking fostered by illustrations in scientific text. *Journal of Educational Psychology, 81*(2), 240–246.

Mayer, R. (1992). Mathematical problem solving: Thinking as based on domain specific knowledge. In R. Mayer (Ed.), *Thinking, problem solving, and cognition* (pp. 455–489). New York, NY: W. H. Freeman & Co.

Mayer, R., & Gallini, J. (1990). When is an illustration worth ten thousand words? *Journal of Educational Psychology, 82*(4), 715–726.

Mendenhall, W., Beaver, R. J., & Beaver, B. M. (2003). *Introduction to probability and statistics* (11th ed.). Belmont, CA: Duxbury.

Molitor, S., Ballstaedt, S. P., & Mandl, H. (1989). Problems in knowledge acquisition from text and pictures. In H. Mandl & J. Levin (Eds.), *Knowledge acquisition from text and pictures* (pp. 3–35). North-Holland: Elsevier Science.

Mosteller, F. (1980). Classroom and platform performance. *The American Statistician, 34*(1), 11–17.

National Council of Teachers of Mathematics (2003). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and the velocity sign. *Journal of Mathematical Behavior, 13*(4), 389–422.

Novick, L. (1990). Representational transfer in problem solving. *Psychological Science, 1*(2), 128–132.

Novick, L. (2002). Spatial diagrams: Key instruments in the toolbox for thought. In D. Medin (Ed.), *The psychology of learning and motivation: Advances in research and theory, 40* (pp. 279–325). San Diego, CA: Academic.

Novick, L., & Hmelo, C. (1994). Transferring symbolic representations across nonisomorphic problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 20*(6), 1296–1321.

Novick, L., & Hurley, S. (2001). To matrix, network, or hierarchy: That is the question. *Cognitive Psychology, 42*(2), 158–216.

Novick, L., Hurley, S., & Francis, M. (1999). Evidence for abstract, schematic knowledge of three spatial diagram representations. *Memory & Cognition, 27*(2), 288–308.

O'Connell, A. A., & Corter, J. E. (1993, April). *Student misconceptions in probability problem-solving*. Paper presented at annual meeting of the American Educational Research Association, Atlanta, GA.

Olive, J. (1998). Opportunities to explore and integrate mathematics with the Geometer's Sketchpad. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 395–418). Mahwah, NJ: Lawrence Erlbaum.

Pantziara, M., Gagatsis, A., & Elia, I. (2009). Using diagrams as tools for the solution of non-routine mathematical problems. *Educational Studies in Mathematics, 72*(1), 39–60.

Penner, D. E., Giles, N. D., Lehrer, R., & Schauble, L. (1996). Building functional models: Designing an elbow. *Journal of Research in Science Teaching, 34*(2), 125–143.

Polya, G. (1957). *How to solve it.* Princeton, NJ: Princeton University Press.

Presmeg, N. C. (1986). Visualization in high school mathematics. *For Learning of Mathematics, 63*(3), 42–46.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In Á. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205–236). Rotterdam: Sense.

Reusser, K. (1996). From cognitive modeling to the design of pedagogical tools. In S. Vosniadou, E. De Corte, R. Glaser, & H. Mandl (Eds.), *International perspectives on the design of technology-supported learning environments* (pp. 81–103). Mahwah, NJ: Lawrence Erlbaum.

Rival, I. (1987). Picture puzzling: mathematicians are rediscovering the power of pictorial reasoning. *The Sciences, 27*, 40–46.

Robinson, K. M. (2001). The validity of verbal reports in children's subtraction. *Journal of Educational Psychology, 93*(1), 211–222.

Ross, B. H. (1984). Remindings and their effects in learning a cognitive skill. *Cognitive Psychology, 16*(3), 371–416.

Ross, B. H. (1989). Remindings in learning and instruction. In S. Vosniadou & A. Ortony (Eds.), *Similarity in analogical reasoning* (pp. 438–469). Cambridge, UK: Cambridge University Press.

Ross B. H., & Kennedy, P. T. (1990). Generalizing from the use of earlier examples in problem solving. *Journal of Experimental Psychology: Learning, Memory, & Cognition, 16*(1), 42–55.

Roth, W. M., & McGinn, M. K. (1998). Inscriptions: Toward a theory of representing social practice. *Review of Educational Research, 68*(1), 35–59.

Russell, W. E. (2000). The use of visual devices in probability problem solving. (Doctoral Dissertation, Columbia University, 2000). *Dissertation Abstracts International, 61*, 1333.

Santos-Trigo, M. (1996). An exploration of strategies used by students to solve problems with multiple ways of solution. *Journal of Mathematical Behavior, 15*(3), 263–284.

Schoenfeld, A. (1994). *Mathematical thinking and problem solving*. Mahwah, NJ: Lawrence Erlbaum.

Schreiber, C. (2004). The interactive development of mathematical inscriptions–A semiotic perspective on pupils' externalisation in an internet chat about mathematical problems. *ZDM, 36*(6), 185–194.

Schwartz, D. L., & Martin, T. (2004). Inventing to prepare for future learning: The hidden efficiency of encouraging original student production in statistics instruction. *Cognition and Instruction, 22*(2), 129–184.

Sedlmeier, P., & Gigerenzer, G. (2001). Teaching Bayesian reasoning in less than two hours. *Journal of Experimental Psychology: General, 130*(3), 380–400.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics, 14*(1), 44–55.

Stylianou, D. A. (2002). On the interaction of visualization and analysis: The negotiation of a visual representation in expert problem solving. *Journal of Mathematical Behavior, 21*(3), 303–317.

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning, 6*(4), 353–387.

Tatsuoka, K. K., Corter, J. E., & Tatsuoka, C. (2004). Patterns of diagnosed mathematical content and process skills in TIMSS-R across a sample of twenty countries. *American Educational Research Journal, 41*(4), 901–926.

Tukey, J. (1977). *Exploratory data analysis*. Boston: Addison-Wesley.

Tversky, B. (2001). Spatial schemas in depictions. In M. Gattis (Ed.), *Spatial schemas and abstract thought* (pp. 79–112). Cambridge, MA: MIT Press.

Uesaka, Y., Manalo, E., & Ichikawa, S. (2007). What kinds of perceptions and daily learning behaviors promote students' use of diagrams in mathematics problem solving? *Learning and Instruction, 17*(3), 322–335.

van Garderen, D., & Montague, M. (2003). Visual-spatial representation, mathematical problem solving, and students of varying abilities. *Learning Disabilities: Research & Practice, 18*(4), 246–254.

Vygotsky, L. S. (1978). *Mind and society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Weaver, J. L., & Quinn, R. L. (1999). Geometer's sketchpad in secondary geometry. *Computers in the Schools, 15*(2), 83–95.

Zahner, D. (2005). Clinical interviewing and problem-solving protocols to uncover the cognitive processes of probability problem solvers. Doctoral dissertation, Columbia University, *Dissertation Abstracts International*, 66, 2851.

Zahner, D., & Corter, J. E. (2002, April). *Clinical interviewing to uncover the cognitive processes of probability problem solvers*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

APPENDIX A:
The Six Problems with Their Variants

Topic	*Typical Variant*	*Atypical Variant*	*Complex Variant*
Joint Events	Twenty percent of the students in a chemistry class are physics majors. Sixty percent of the students are chemistry majors. Twelve percent of them are double majoring in chemistry and physics. What is the probability that a randomly chosen student will be a chemistry major or a physics major (or both)?	Twenty percent of the students in a chemistry class are physics majors. Sixty percent of the students in the class are chemistry majors. Sixty-eight percent of the students are majoring in either chemistry or physics. What is the probability that a randomly selected student is double-majoring in chemistry and physics?	Twenty percent of the students in a chemistry class are physics majors. Sixty percent of the students in the class are chemistry majors and five percents are math majors. Of all of the students, twelve percent of them are double majoring in chemistry and physics, eight percent of them are double majoring in chemistry and math, and three percent of them are double majoring in physics and math. Finally, one percent of the students are crazy enough to triple major in all three subjects. What is the probability that a randomly chosen student will be majoring in chemistry or physics or math (or in some double or triple major involving these three subjects)?
Conditional Probability	It's election time! Candidate H is running for mayor. Forty-five percent of the population in the city is Republican. Fifty-five percent of the population is Democrat. Given that a voter is a Republican, there is a 25% chance that he/she will vote for H. If the voter is a Democrat, there is an 85% chance that he/she will vote for H. What is the probability that a randomly selected voter will vote for Candidate H?	It's election time! Candidate H is running for mayor. Forty-five percent of the population is Republican. Fifty-five percent of the population is Democrat. The probability that a voter will vote for H, given that he/she is a Democrat is 0.85. What is the probability that a randomly chosen voter will be a Democrat and vote for H?	It's election time! Candidate H is running for mayor. Forty percent of the population in the city is Republican. Fifty percent of the population is Democrat and ten percent of the population is registered as Independent. Given that a voter is a Republican, there is a 25% chance that he/she will vote for H. If the voter is a Democrat, that person has an 85% probability of voting for H. If the voter is an Independent, that person has a 2% chance of voting for H. What is the probability that a randomly selected voter will vote for Candidate H?

Independent Events	Company 1 and Company 2 manufacture doo-dads. The probability that Company 1 manufactures a defective doo-dad is .06. The probability that Company 2 manufactures a defective doo-dad is .10. You randomly select one doo-dad from each company. What is the probability that both doo-dads will be defective?	Company 1 and Company 2 manufacture doo-dads. The probability that a randomly selected doo-dad from C1 is defective is .05. The probability that two randomly selected doo-dads (one from each company) are both defective is .006. What is the probability that a randomly selected doo-dad from C2 is defective?	Company 1, Company 2 and Company 3 manufacture doo-dads. The probability that C1 manufactures a defective doo-dad is .06. The probability that C2 manufactures a defective doo-dad is .10. The probability that C3 manufactures a defective doo-dad is .01. You randomly choose a doo-dad from each company. What is the probability that you will get defective doo-dads from all three companies?
Permutations	Four job candidates, Mr. A, Ms. B, Dr. C, and Mr. D, are interviewed then ranked by the interviewer. What is the probability that they wind up ranked D-C-B-A?	Four job candidates, Mr. A, Ms. B, Dr. C, and Mr. D, are interviewed, then ranked by the interviewer. What is the probability that they wind up ranked with Mr. D first and Dr. C second?	Six job candidates, Mr. A, Ms. B, Dr. C, Mr. D, Ms. E and Fraulein F, are interviewed, then ranked by the interviewer. What is the probability that they wind up ranked F-E-D-C-B-A?
Fundamental Principle	Colin can take any one of four roads to get to work. How many different ways can he get to work and then home again?	Colin can take any one of four roads to get to work. How many different ways can he get to work and then home again if he cannot take the same road twice?	Colin takes any one of four roads to get to and from work, any of three roads to get from work to the gym, and any of five roads home from the gym. If he goes from home to work to the gym and then home today, how many different routes can he take?
Combinations	Michelle is packing CDs to listen to in her car on the way home from work. There are 8 CDs in her collection. If she randomly grabs 3, how many possible outcomes are there?	Michelle is packing CDs to listen to in her car on the way home from work. There are 8 CDs in her collection, five of which are classical. She randomly grabs 3 CDs to listen to. How many outcomes are there that consist of all classical CDs?	Michelle is packing CDs to listen to in her car on the way home from work. There are 8 CDs in her collection, five of which are classical. She randomly grabs 3 CDs to listen to. What is the probability that the three CDs she winds up with are all classical?

APPENDIX B: Schematic Formula-based Solutions for Typical, Atypical, and Complex Variants of Each Problem Topic

Problem Topic	*Typical*	*Atypical*	*Complex*
P1 Joint Events	$P(P \cup C) = P(P) + P(C) + P(P \cap C)$	$P(P \cap C) = P(P \cup C) - [P(P) + P(C)]$	$P(P \cup C \cup M) = P(P) + P(C) + P(M) - P(P \cap C) - P(P \cap M) - P(C \cap M) + P(P \cap C \cap M)$
P2 Conditional Probability	$P(H) = P(R \cap H) + P(D \cap H)$ $P(R \cap H) = P(R \mid H) * P(R)$ $P(D \cap H) = P(H \mid D) * P(D)$	$P(D \cap H) = P(H \mid D) * P(D)$	$P(H) = P(R \cap H) + P(D \cap H) + P(I \cap H)$ $P(R \cap H) = P(R \mid H) * P(R)$ $P(D \cap H) = P(H \mid D) * P(D)$ $P(I \cap H) = P(H \mid I) * P(I)$
P3 Independent Events	$P(C1 \cap C2) = P(C1) * P(C2)$	$P(C2) = \frac{P(C1 \cap C2)}{P(C1)}$	$P(C1 \cap C2 \cap C3) = P(C1) * P(C2) * P(C3)$
P4 Permutations	# outcomes = n! n! = (n)(n-1). . .(2)(1) where n = 4	# outcomes = (n)(n-1) where n = 4	# outcomes = n! n! = (n)(n-1). . .(2)(1) where n = 6
P5 Fundamental Principle	# outcomes = n*n where n = 4	# outcomes = n*(n-1) where n = 4	# outcomes = n1*n2*n3 where n1 = 4, n2 = 3, n3 = 5
P6 Combinations	${}_{n}C_{k} = \frac{n!}{k!(n-k)!}$ where n = 8; k = 3	${}_{m}C_{k} = \frac{m!}{k!(m-k)!}$ where m = n – 3 = 5; k = 3	$P(A) = \frac{{}_{m}C_{k}}{{}_{n}C_{k}} = \frac{\frac{m!}{k!(m-k)!}}{\frac{n!}{k!(n-k)!}}$