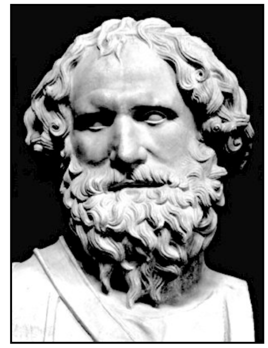
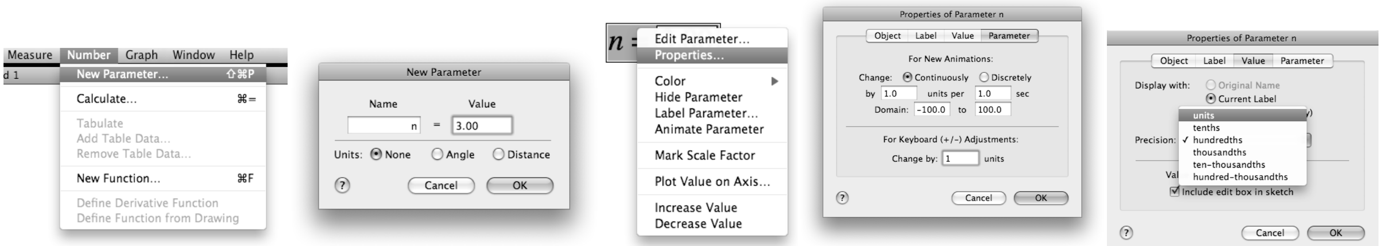
An Archimedean Walk



Archimedes is a well-known Greek mathematician, born in 287 BC. Building on many of Euclid's theorems and postulates of Geometry, Archimedes brought the idea of *iteration* to studying particular geometrical problems. It was well known at the time that the larger a circle was, the larger its circumference. Archimedes began investigating the relationship between these two things by first replicating an experiment with shapes he could know this answer to, regular polygons. In the Sketchpad activity and questions that follow, you will walk Archimedes' walk and talk Archimedes' talk.

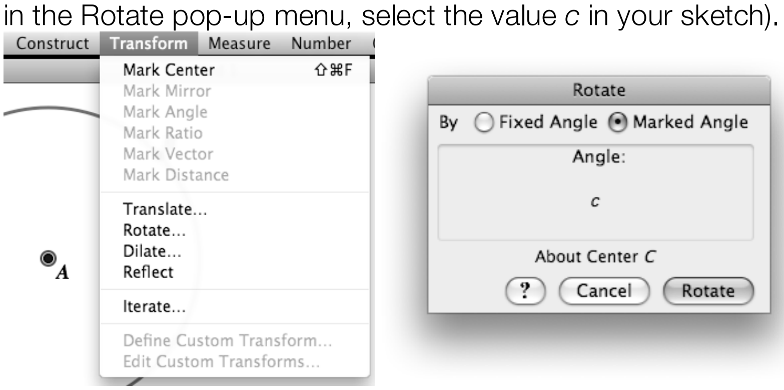
Modeling in Sketchpad

1. Create a **New Parameter** (under the Number menu) and label it *n*. The starting value should be 3. Under the **Properties** of the Parameter, make the **Keyboard Adjustment** 1 unit; change the Value **Precision** to Units.

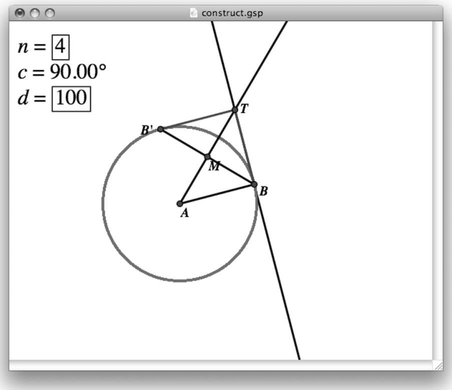


2. Create a new calculation (Calculate under the Number menu) of 360°/*n*. While in the Calculate pop-up menu, you can select *n* from your sketch. After you have created this new calculation, change the Label to *c*. Create a **New Parameter** (under the Number menu) named *d* with a value of 100.

3. Construct a large circle and label the center as *A* and the point on the circle as *B*. Select *A* and **Mark as Center** (under the Transform menu). Select point *B* and **Rotate** (under the Transform menu) by *c* degrees (while in the Rotate pop-up menu, select the value *c* in your sketch).

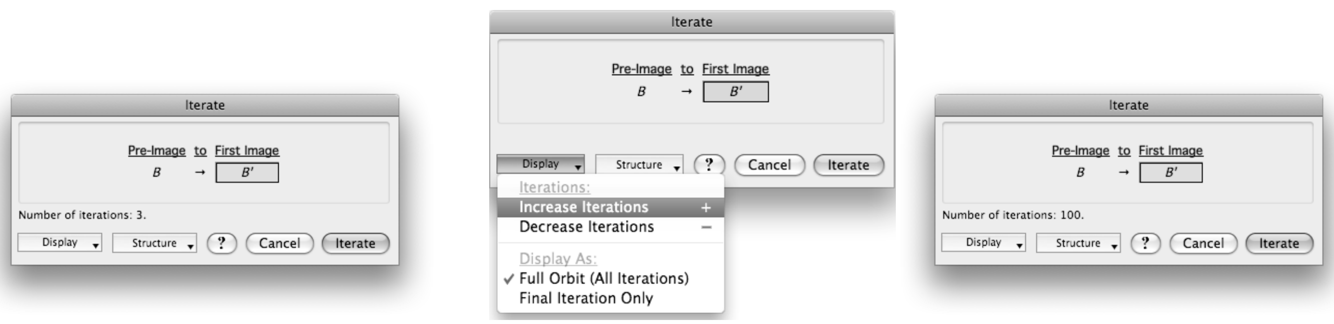


4. Construct a line segment between *B* and *B'*. Construct the midpoint, M, of $\overline{BB'}$. Construct Ray $\overrightarrow{AM}$. Construct radius $\overline{AB}$. Construct a line, *m*, perpendicular to $\overline{AB}$ through point B. Label the intersection of Ray $\overrightarrow{AM}$ and *line m*, point T. Construct segments $\overline{TB}$ and $\overline{TB'}$.



Hide Ray $\overrightarrow{AM}$, radius $\overline{AB}$, *line m*, and midpoint M (At this point you may desire to change $\overline{TB}$ and $\overline{TB'}$ to a different color than $\overline{BB'}$ to distinguish between the inscribed and circumscribed polygons)

5. Deselect everything. Then select point B and the parameter *d* in that order, and **Iterate to Depth** (under the Transform menu, hold the Shift key and Iterate will change to Iterate to Depth). While the Iterate pop-up menu is open, select the point *B'* on the circle in your sketch. It should show the number of iterations to be 100 (e.g. the parameter *d*). Then press **Iterate** (you can also change the number of iterations to be 100 by pressing Shift + if the Iterate to Depth command is unfamiliar).



Questions for Reflection

Q1. Describe what happens. Change the value of *n* by typing Shift +. What shape(s) are created? What is the interpretation of *n*?

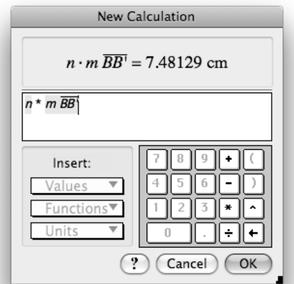
Q2. What does the calculation 360°/*n* find? Explain why this helps construct a regular n-gon.

Q3. How would we describe the inside *n*-gon and the outside *n*-gon in relation to the circle? Order these from least to greatest: the circumference of the circle, the perimeter of the inscribed and circumscribed regular polygons.

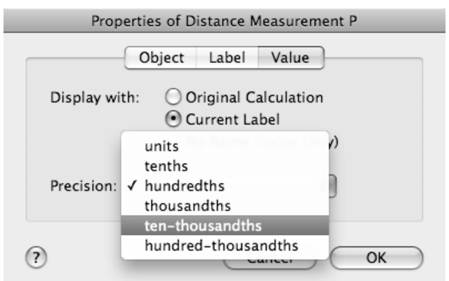
Q4. What are the properties of regular polygons that were used to circumscribe the circle? How were these used in the construction process?

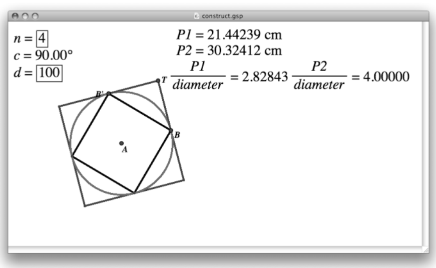
6. Select points *A* and *B*, and **Measure** the **Distance** (re-label this *r*, the radius). Create a new calculation to multiply *r* by 2 to find the diameter, and label it *diameter*.

7. Find the perimeter of the inscribed polygon by measuring the **Length** of $\overline{BB'}$ (under the Measure menu). Create a new calculation, and **Calculate** *n* times $\overline{BB'}$ by clicking on the values in the sketch. Label this *P1* for perimeter. Find the perimeter of the circumscribed polygon by measuring the **Length** of $\overline{TB}$. Create a new calculation, and **Calculate** 2**n* times $\overline{TB}$ by clicking on the values in the sketch. Label this *P2*.



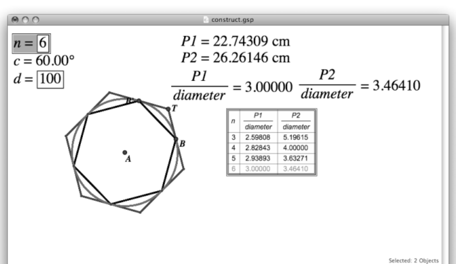
8. **Calculate** the ratio *P1/diameter*. Under the **Properties** of the calculation, change the Value Precision to hundred-thousandths (i.e. 5 decimal places). Calculate the ratio *P2/diameter*. Hide any unnecessary measurements like *r, d,* $\overline{BB'}$, $\overline{TB}$.





9. Select *n*, *P1/diameter* and *P2/diameter* in that order, and **Tabulate** (under the Number menu).

n	*P1* / *diameter*	*P2* / *diameter*
3	2.59808	5.19615
4	2.82843	4.00000
5	2.93893	3.63271
6	3.00000	3.46410



10. Select both *n* and the Table. Now increase the values of *n* by typing "Shift +." Watch as the *n*-gons increase in the number of sides, and the table of values records each one. Make sure to use the values of n=6, 12, 24, 48, 96

Questions for Reflection:

Use your sketch to adjust the values of *n*, to select and drag points, etc. to answer the following questions.

Q3. As $n \to \infty^+$, what shape do both the polygons resemble? Which is bigger?

Q4. When you change *d* (by dragging Point B), what do you observe about the ratio *P1/diameter*? *P2/diameter*?

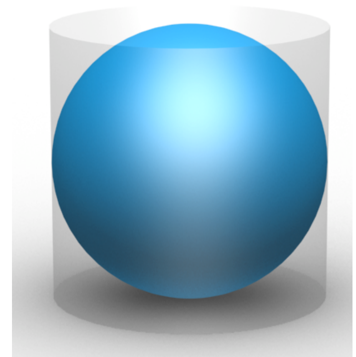
Q5. What do you observe about *P1/diameter* and *P2/diameter* as *n* increases? What are the values when *n*=96*?

Q6. What value does it look like both *P1/diameter* and *P2/diameter* are approaching? What did Archimedes conclude about the ratio of the Circumference to the diameter of a circle? How accurately did Archimedes approximate π?

Extension Activity

One extension to this activity that makes the idea of upper and lower bounds clearer is plotting values on a graph (Figure 9). The visual effect of watching both of these numbers converge to π on a graph is yet another method to help convince students of the validity of Archimedes method in approximating π. To do so: Start with *n*=3. Select (in order) *n* and *P1/d*iameter, and `Plot as (x,y)` (Under the Graph menu). Select (in order) *n* and *P2/d*iameter, and `Plot as (x,y)` (Under the Graph menu). Select both of these points and `Trace Plotted Points` (under the Display menu). As you increase the values of *n*, you can watch the points simultaneously close in on π from above and below.

“Sketching Up” with Archimedes



The achievement of which Archimedes was most proud happened to involve understanding the sphere. His use of limits for larger and larger regular n-gons helped him find upper and lower bounds for π. Using some similar ideas, Archimedes was able to conclude the following for a sphere:

> A sphere has 2/3 the volume and surface area of its circumscribing cylinder.

As a matter of fact, a sphere and cylinder were placed on the tomb of Archimedes at his request. In the SketchUp activity and questions that follow, you will explore some of his methods and reasoning while using SketchUp in approximating these values.

Modeling in SketchUp

1. Using the Circle Tool, create a circle centered at the origin with a radius of 2 meters. In the lower right hand corner, you can adjust the radius so that it is exactly 2 meters. (NOTE: when using the circle tool, you can adjust the “number of sides” that will be used to model a circle – the default in SketchUp is 24 sides, which we will use in this activity.)

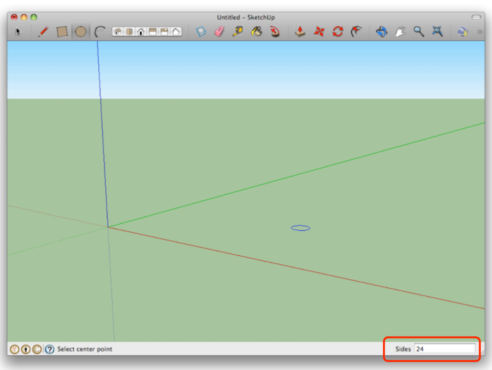


Figure 1

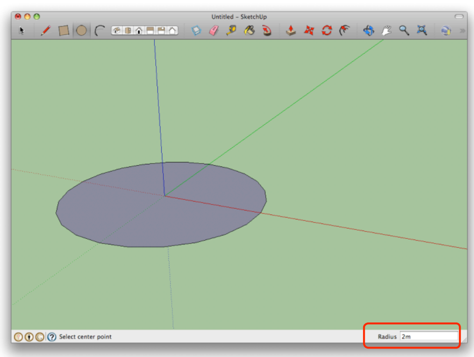


Figure 2

2. Using the pencil tool, create a diameter of the circle. Count two vertices away from one endpoint of the diameter (on the circular 24-gon) and create a chord parallel to the diameter using the pencil tool (the endpoint should be another point on the circular 24-gon). Create another chord on the other half of the circle in the same way (Figure 3).
3. Connect the endpoints of each of the three chords to create a concave octagon (Figure 4).

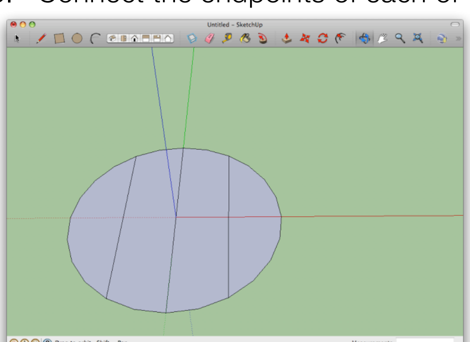


Figure 3

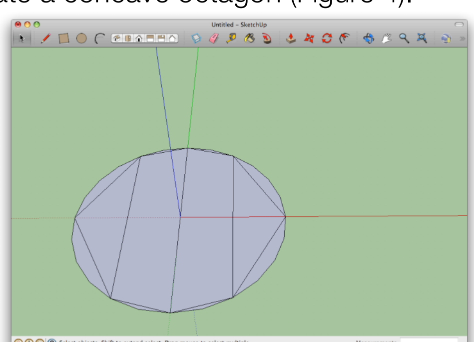


Figure 4

“Sketching Up” with Archimedes

Questions for Reflection:
Q1. Is the octagon regular?

Q2. Using two chords, you can create a regular hexagon. Explain how this justifies that the diameter perpendicular to the constructed diameter is partitioned into four equal lengths (i.e. that the distance between the three parallel chords is equal, and also equal to the distance from the outer chords to the end of the circle).

4. Construct another, smaller circle along your diameter, centered at the origin that is perpendicular to the plane of the current circle (Figure 5).
5. Use the arrow tool to select the outer edge of the smaller circle. You will need to hold down the “Shift” key to select both the Top and Bottom halves of the circle. Select the “Follow me” tool , and click the region adjacent to the diameter (Figure 6). **Q3:** What 3-d shape is this?

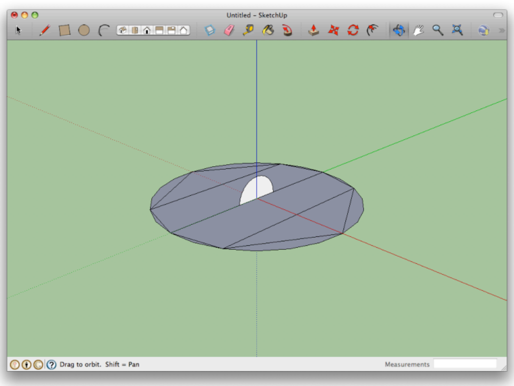


Figure 5

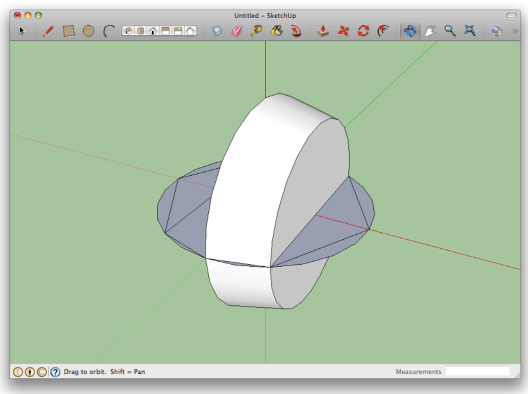


Figure 6

6. Select the Top and Bottom halves of the circle again, and use the follow me tool to revolve the triangular region around the axis (Figure 7). **Q4:** What 3-d shape is this?
7. Do this two more times to create a complete model of the shape. (Hint: you will need to revolve the triangular part first so that you can select the circle the last time to revolve to region next to the diameter). (Figure 8).

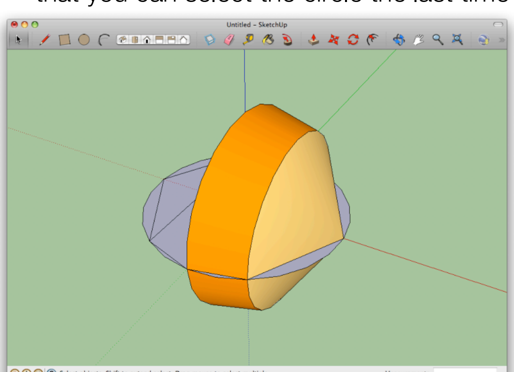


Figure 7

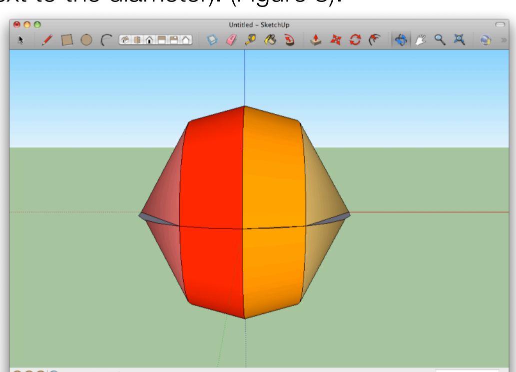


Figure 8

“Sketching Up” with Archimedes

Questions for Reflection:

Q5. Archimedes had access to upper and lower bounds for $\sqrt{3}$. How did using a hexagon help?

Q6. What two shapes did Archimedes have to be able to calculate the SA and Volume for to create his model of a sphere.

Q7. Archimedes began by using equal distances between the lines. What did this mean about the heights, h, of each of the different shapes used to model the sphere?

8. Given any even-sided regular polygon (we will use a regular 24-gon), you can partition the shape using parallel chords so that revolving each region around the axis results in a frustrum or a cone (Figure 9). In this case, the heights of the frustrums/cones decrease; however, given any regular n-gon, with a central angle, $\vartheta = 360/n^{\circ}$, you can determine the height of each subsequent frustrum/cone (look at Figure 10). It was this realization that allowed Archimedes to find determine the SA and Volume of larger n-gons, ultimately resulting in an accurate representation of a sphere.

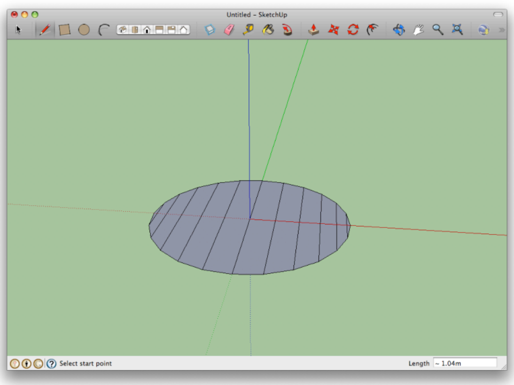


Figure 9

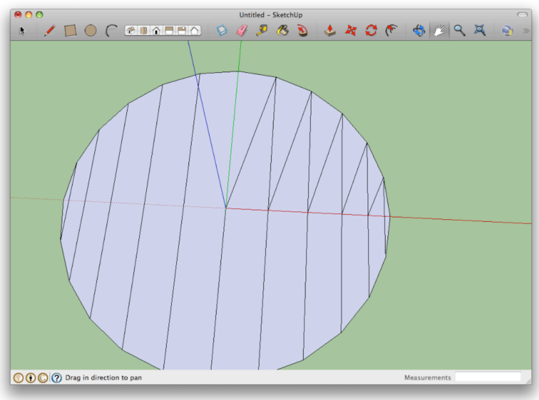


Figure 10

In conclusion, Archimedes was able to compare the SA and Volume of a sphere (a regular n-gon, as $n \rightarrow \infty$) to that of its circumscribing cylinder.

Try to create a sphere in SketchUp. Try to create its circumscribing cylinder.