Systems-level Content Development:

Establishing Learning Progressions

RME Research to Practice Conference

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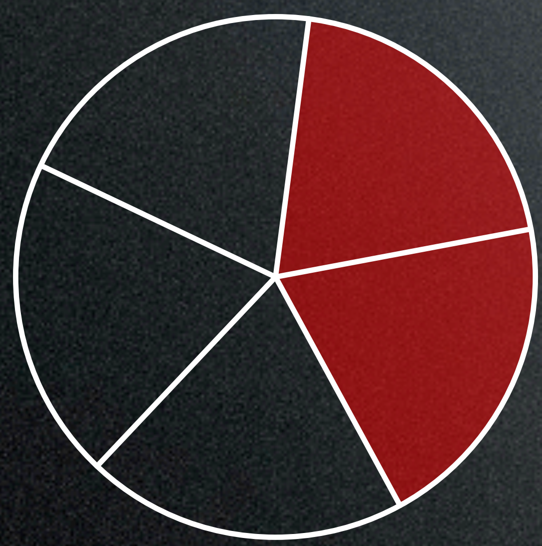
24 February 2012

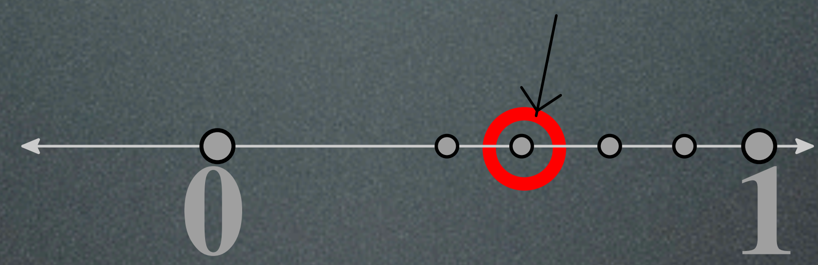
A Math Question

Which of the following are correct representations of 2/5?

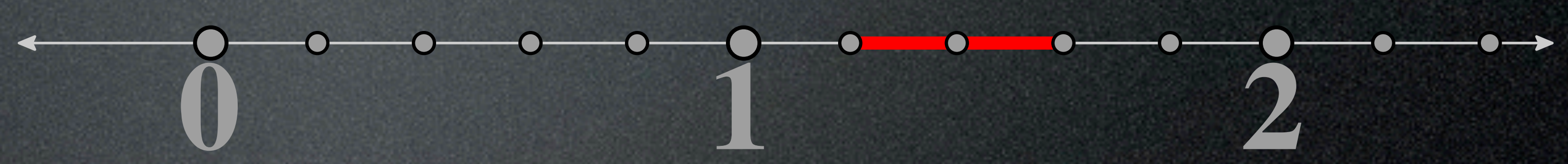
I.



II.



III.



A. I, III only

B. I only

C. II only

D. I, II, III

What are Learning Progressions?

STD or EC Code	Eligible Content	Grades										Algebra I		Algebra II		Geometry	
		K	1	2	3	4	5	6	7	8	HS	**Module 1** Operations and Linear Functions & Inequalities	**Module 2** Linear Functions and Data Organizations	**Module 1** Numbers Systems and Data Analysis	**Module 2** Non-Linear Expressions and Equations	**Module 1** Geometric Properties and Relations	**Module 2** Geometrical Reasoning
Numbers and Operations: Operations on numeric and symbolic expressions																	
M3.A.1.1.1	Match the word name with the appropriate whole number (up through 9,999).																
M3.A.1.1.2	Differentiate between and/or give examples of even and odd number (limit to 3 digits).																
M3.A.1.3.1	Count a collection of bills and coins less than $5.00 (penny, nickel, dime, quarter, dollar). Money may be represented as 15 cents, 15¢ or $0.15.																
M3.A.1.3.3	Make change for an amount up to $5.00 with no more than $2.00 change given (penny, nickel, dime, quarter, dollar).																
M4.A.1.1.1	Write the fraction or decimal, including mixed numbers, which corresponds to a drawing or set—no simplification necessary.																
M4.A.1.1.3	Match the standard number form to the word form of decimal numbers (through the tenths place).																
M4.A.1.1.4	Write whole numbers in expanded, standard and/or word form through 6 digits (example of standard to expanded form: 43,076 = 40,000+3000+70+6).																
M4.A.1.2.1	Locate/identify fractions or decimals on a number line (decimals and fractions through the tenths—do not mix fractions and decimals).																
M5.A.1.2.1	Match the standard form to the word form of decimal numbers through the hundredths.																
M5.A.1.2.2	Identify the place value of a digit (from millions through hundredths).																
M5.A.1.4.1	Locate/identify integers on a number line (greater than or equal to -20).																
M5.A.3.2.1	Use addition, subtraction, multiplication and division to compute accurately without a calculator (multipliers up to 2 digits, single-digit divisors or multiples of 10—whole numbers through thousands and decimals through hundredths—no divisors with decimals).																
M6.A.1.3.1	Find the Greatest Common Factor (GCF) of two numbers (through 50) and/or use the GCF to simplify fractions.																
M6.A.1.3.2	Find the Least Common Multiple (LCM) of two numbers (through 50) and/or use the LCM to find the common denominator of two fractions.																
M6.A.1.3.3	Use divisibility rules for 2, 3, 5 and/or 10 to draw conclusions and/or solve problems.																
M7.A.1.1.1	Convert between fractions, decimals and/or percents (e.g., 20% = 0.2 = 1/5) (terminating decimals only).																
M7.A.1.2.2	Locate/identify decimals, fractions, mixed numbers and/or integers on a number line (a mix of these number forms may be on the same number line).																
M7.A.2.1.1	Use the order of operations to simplify numerical expressions (may use parentheses, brackets, +, -, x, ÷, squares up to 10² and cubes up to 4³; whole numbers only).																
[illegible]	Calculate and/or apply unit rates or unit prices (terminating decimals through the hundredth place only).																

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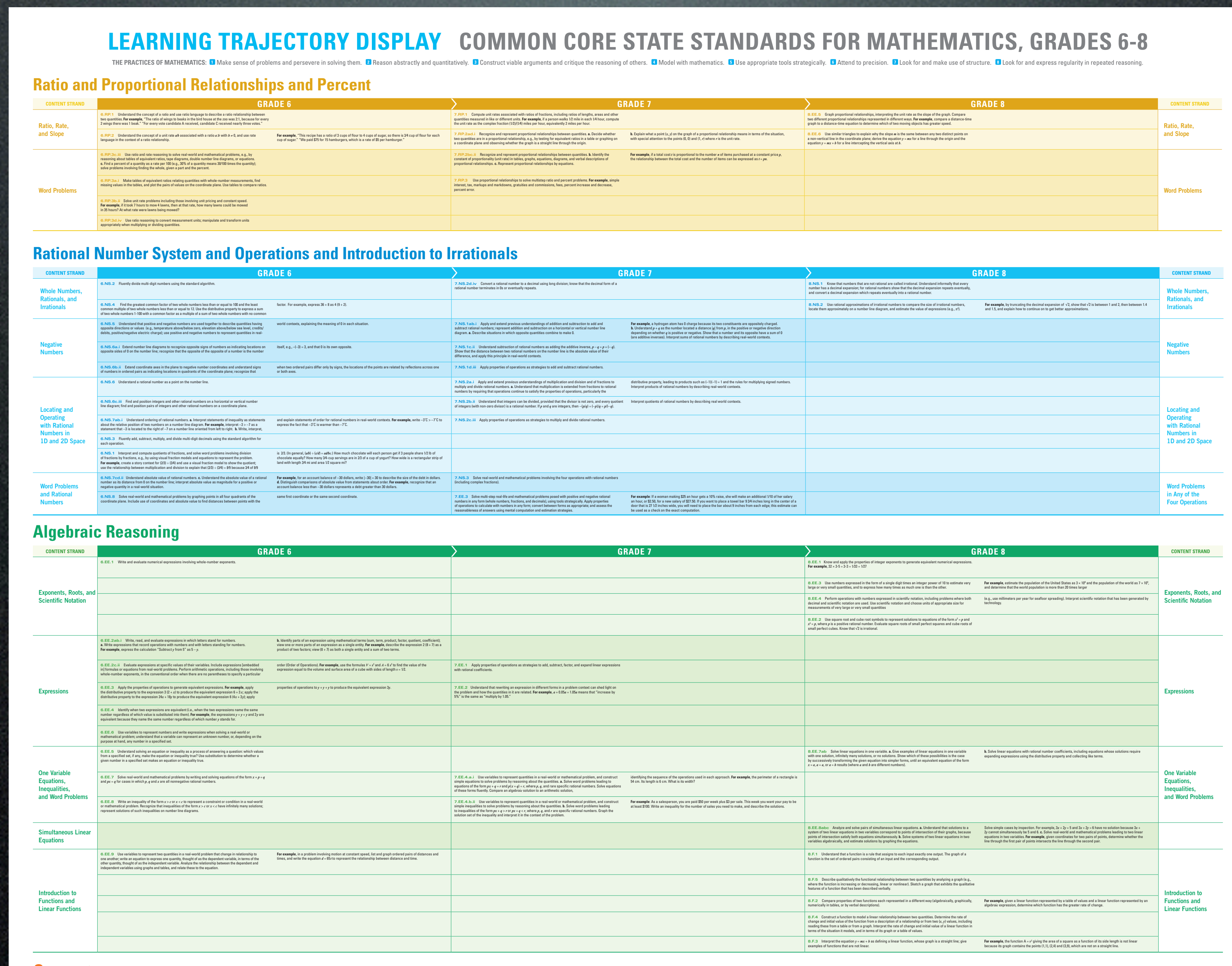

What are Learning Progressions?

Counting and Cardinality

Several progressions originate in knowing number names and the count sequence:[K.CC.1]

From saying the counting words to counting out objects Students usually know or can learn to say the counting words up to a given number before they can use these numbers to count objects or to tell the number of objects. Students become fluent in saying the count sequence so that they have enough attention to focus on the pairings involved in counting objects. To count a group of objects, they pair each word said with one object.[K.CC.4a] This is usually facilitated by an indicating act (such as pointing to objects or moving them) that keeps each word said in time paired to one and only one object located in space. Counting objects arranged in a line is easiest; with more practice, students learn to count objects in more difficult arrangements, such as rectangular arrays (they need to ensure they reach every row or column and do not repeat rows or columns); circles (they need to stop just before the object they started with); and scattered configurations (they need to make a single path through all of the objects).[K.CC.5] Later, students can count out a given number of objects,[K.CC.5] which is more difficult than just counting that many objects, because counting must be fluent enough for the student to have enough attention to remember the number of objects that is being counted out.

From subitizing to single-digit arithmetic fluency Students come to quickly recognize the cardinalities of small groups without having to count the objects; this is called *perceptual subitizing*. Perceptual subitizing develops into *conceptual subitizing*—recognizing that a collection of objects is composed of two subcollections and quickly combining their cardinalities to find the cardinality of the collection (e.g., seeing a set as two subsets of cardinality 2 and saying "four"). Use of conceptual subitizing in adding and subtracting small numbers progresses to supporting steps of more advanced methods for adding, subtracting, multiplying, and dividing single-digit numbers (in several OA standards from Grade 1 to 3 that culminate in single-digit fluency).

From counting to counting on Students understand that the last number name said in counting tells the number of objects counted.[K.CC.4b] Prior to reaching this understanding, a student who is asked "How many kittens?" may regard the counting performance itself as the answer, instead of answering with the cardinality of the set. Experience with counting allows students to discuss and come to understand the second part of K.CC.4b—that the number of objects is the same regardless of their arrangement or the order in which they were counted. This connection will continue in Grade 1 with the

K.CC.1 Count to 100 by ones and by tens.

K.CC.4a Understand the relationship between numbers and quantities; connect counting to cardinality.

a When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

K.CC.4b Understand the relationship between numbers and quantities; connect counting to cardinality.

b Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

What are Learning Progressions?

Developmental Levels for Recognizing Number and Subitizing (Instantly Recognizing)

Age Range	Level Name	Level	Description
2	Small Collection Namer	1	Names groups of one to two, sometimes three. For example, shown a pair of shoes, child says "Two shoes."
3	Maker of Small Collections	2	Nonverbally makes a small collection (no more than 4, usually 1-3) with the same number another collection. For example, when shown a collection of 3, makes another collection of 3.
4	Perceptual Subitizer to 4	3	Instantly recognizes collections up to 4 when briefly shown and verbally names the number of items. For example, when shown 4 objects briefly, says "four."
5	Perceptual Subitizer to 5	4	Instantly recognize briefly shown collections up to 5 and verbally name the number of items. For example, when shown 5 objects briefly, says "5."
5	Conceptual Subitizer to 5+	5	Verbally labels all arrangements to about 5, when shown only briefly. For example, says "Five! Why? Because I saw three and two and so I said five."

Age Range	Level Name	Level	Description
5	Conceptual Subitizer to 10	6	Verbally label most briefly shown arrangements to 6, then up to 10, using groups. For example, says, "In my mind, I made two groups of 3 and one more, so 7."
6	Conceptual Subitizer to 20	7	Verbally label structured arrangements up to 20, shown only briefly, using groups. For example, says, "I saw three fives, so 5, 10, 15."
7	Conceptual Subitizer with Place Value and Skip Counting	8	Verbally label structured arrangements shown only briefly, using groups, skip counting, and place value. For example, says, "I saw groups of ten and twos, so 10, 20, 30, 40, 42, 44, 46...46!"
8+	Conceptual Subitizer with Place Value and Multiplication	9	Verbally label structured arrangements shown only briefly, using groups, multiplication, and place value. For example, says, "I saw groups of ten and threes so I thought, five tens is 50 and four threes is 12, so 62 in all."

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4. Learning Performances at each Level that articulate students' performance capability
5. Assessments that measure student development along the progression

A Science example

Solar System Progression:

from Wilson (2009)

Level	Description
5 8th grade	Student is able to put the motions of the Earth and Moon into a complete description of motion in the Solar System which explains: • the day/night cycle • the phases of the Moon (including the illumination of the Moon by the Sun) • the seasons
4 5th grade	Student is able to coordinate apparent and actual motion of objects in the sky. Student knows that • the Earth is both orbiting the Sun and rotating on its axis • the Earth orbits the Sun once per year • the Earth rotates on its axis once per day, causing the day/night cycle and the appearance that the Sun moves across the sky • the Moon orbits the Earth once every 28 days, producing the phases of the Moon COMMON ERROR: Seasons are caused by the changing distance between the Earth and Sun. COMMON ERROR: The phases of the Moon are caused by a shadow of the planets, the Sun, or the Earth falling on the Moon.
3	Student knows that: • the Earth orbits the Sun • the Moon orbits the Earth • the Earth rotates on its axis However, student has not put this knowledge together with an understanding of apparent motion to form explanations and may not recognize that the Earth is both rotating and orbiting simultaneously. COMMON ERROR: It gets dark at night because the Earth goes around the Sun once a day.
2	Student recognizes that: • the Sun appears to move across the sky every day • the observable shape of the Moon changes every 28 days Student may believe that the Sun moves around the Earth. COMMON ERROR: All motion in the sky is due to the Earth spinning on its axis. COMMON ERROR: The Sun travels around the Earth. COMMON ERROR: It gets dark at night because the Sun goes around the Earth once a day. COMMON ERROR: The Earth is the center of the universe.
1	Student does not recognize the systematic nature of the appearance of objects in the sky. Students may not recognize that the Earth is spherical. COMMON ERROR: It gets dark at night because something (e.g., clouds, the atmosphere, "darkness") covers the Sun. COMMON ERROR: The phases of the Moon are caused by clouds covering the Moon. COMMON ERROR: The Sun goes below the Earth at night.
0	No evidence or off-track

A Math example

Equipartitioning:

Important for rational number & fraction development

Case	Equipartitioning Progress Variable
D	1.8 m objects shared among p people, $m > p$
C	1.7 m objects shared among p people, $p > m$
B	1.6 Splitting a continuous whole object into odd # of parts ($n > 3$)
B	1.5 Splitting a continuous whole object among $2n$ people, $n > 2$, & $2n \neq 2^i$
B	1.4 Splitting continuous whole objects into three parts
B	1.3 Splitting continuous whole objects into 2^n shares, with $n > 1$
A	1.2 Dealing discrete items among $p = 3 - 5$ people, with no remainder; mn objects, $n = 3, 4, 5$
A, B	1.1 Partitioning using 2-split (continuous and discrete quantities)

from Mojica & Confrey (2009)

MStar Goal

- Create a Diagnostic Assessment for struggling learners
- Develop and Use Learning Progressions as the framework for Diagnostic
- Better understand “why” students struggle, not “what” they struggle with
- Some of the issues

Your Turn

Learning Goal:

For students to be able to represent a variety of number patterns with tables, graphs, words, and symbolic rules

Your Turn

BELOW PROFICIENCY			PROFICIENT	ADVANCED
Less Complex	⟷	More Complex		
The student will:	**The student will:**	**The student will:**	**The student will:**	**The student will:**
• Determine the next 3 values in a given sequence of numbers (e.g., given the sequence "3, 7, 11, 15 …" conclude that the next three values will be 19, 23, and 27). ------------------------	• Organize the values in a given sequence using a table and/or graph (e.g., where "x-value" represents the placement in the sequence (i.e., 1 for the 1st term, 2 for the 2nd term, etc.) and the y-value represents the value of the term). [NOTE: Include different kinds of patterns, such as numerical, spatial, and recursive.]	• Organize the values in a given sequence using a table and/or graph and determine the recursive pattern in the sequence (e.g., given the sequence "3, 7, 11, 15 …" conclude that the next number is obtained by adding 4 to the previous value) ------------------------	• Organize the values in a given sequence using a table and/or graph and be able to state an explicit rule to find the value of the nth term either symbolically or verbally (e.g., given the sequence "3, 7, 11, 15 …" conclude that the rule is y=4x-1, or an equivalent form, or verbally describing that you have to multiply the term number by 4 and then subtract 1).	• Explain how a table of values can be used to determine whether a function is linear or nonlinear. Explanation should include an example to demonstrate each. ------------------------

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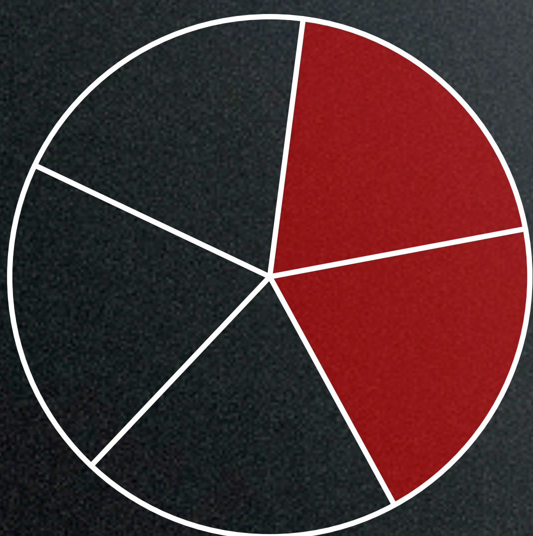

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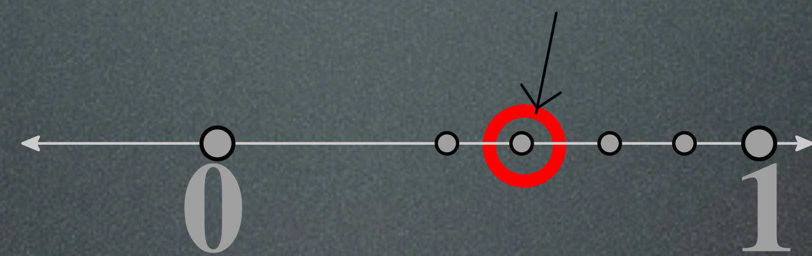

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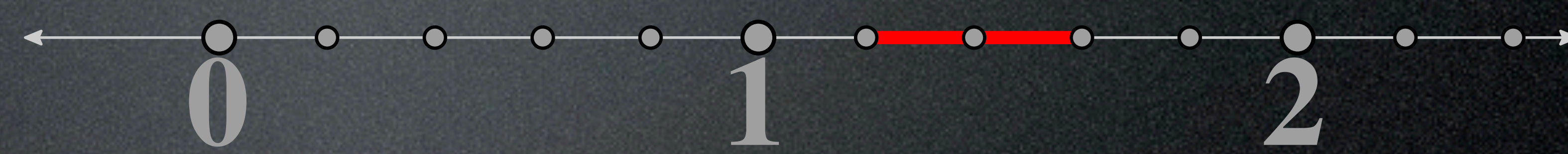

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1.2	**ii.** The student understands that a number has a specific location on the number line based on what is "next" in the list of numbers (ordinal), and that numbers represent a distance or quantity from 0 (cardinal). **(M)** Understands the end point as the distance, regardless of the beginning point **iv.** The student understands the magnitude of "common" fractions (e.g. 1/2, 1/4), and use "common" fractions to estimate magnitude or distance.

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2.2	**i.** The student will be able to partition **shapes** into equal regions (with equal areas) using paper strips and pictorial representations. The student recognizes that shapes of different sizes can be partitioned equally and still represent unit fractions. **(M)** Does not recognize that for fraction models involving area, two parts may look different but hold the same relationship to the whole **ii.** The student makes the connection that a whole is composed of 2 halves, 3 thirds, 4 fourths, and that the number of parts is the denominator of the unit fraction. The student can label each part of the partitioned whole as a fraction.

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1.3	**iv.a.** The student understands the magnitude or distance between two numbers is related to counting the number of equal intervals. **(M)** Views fractions only as part:whole relationships and not as numbers in their own right (e.g. they view 1/4 in relation to 1, but not as its own number, 1/4). **(E)** Incorrectly "counts" intervals between 2/5 and 6/5 as "4." **iv.b.** The student understands that fractions, 1/b, are located by dividing 1 into b equal intervals (e.g. 1/4 as dividing 1 into 4 equal intervals). The student will be able to make the connection that if the numerator is larger than the denominator then that improper fraction is greater than 1, and if the numerator is smaller than the denominator then that fraction is less than 1. (i.e. 3/3=1, so 5/3>1 and 3/5<1). **(M)** Does not grasp that fractions are a quantity (cardinal), measured as a distance from 0.

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1.4	**iv.a.** The student understands the magnitude or distance between two numbers is related to counting the number of equal intervals, including fractional intervals. (e.g. correctly "counts" intervals between 2/5 and 6/5 as "4/5") **iv.b.** The student will be able to partition the number line between 0 and 1 into b equal intervals, and recognizes that each interval is the same fractional unit size, 1/b. They can locate the number 7/4 as the distance of seven 1/4 intervals from 0.

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1.3	**iv.a.** The student understands the magnitude or distance between two numbers is related to counting the number of equal intervals. **(M)** Views fractions only as part:whole relationships and not as numbers in their own right (e.g. they view 1/4 in relation to 1, but not as its own number, 1/4). **(E)** Incorrectly "counts" intervals between 2/5 and 6/5 as "4." **iv.b.** The student understands that fractions, 1/b, are located by dividing 1 into b equal intervals (e.g. 1/4 as dividing 1 into 4 equal intervals). The student will be able to make the connection that if the numerator is larger than the denominator then that improper fraction is greater than 1, and if the numerator is smaller than the denominator then that fraction is less than 1. (i.e. 3/3=1, so 5/3>1 and 3/5<1). **(M)** Does not grasp that fractions are a quantity (cardinal), measured as a distance from 0.
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C

How do LPs help?

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Theoretical Distribution

Grade 5 mathematics

4.

47)1325

a. 28 R 9
b. 28 R 1
c. 30 R 15
d. 28 R 19

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$47\overline{)1325}$

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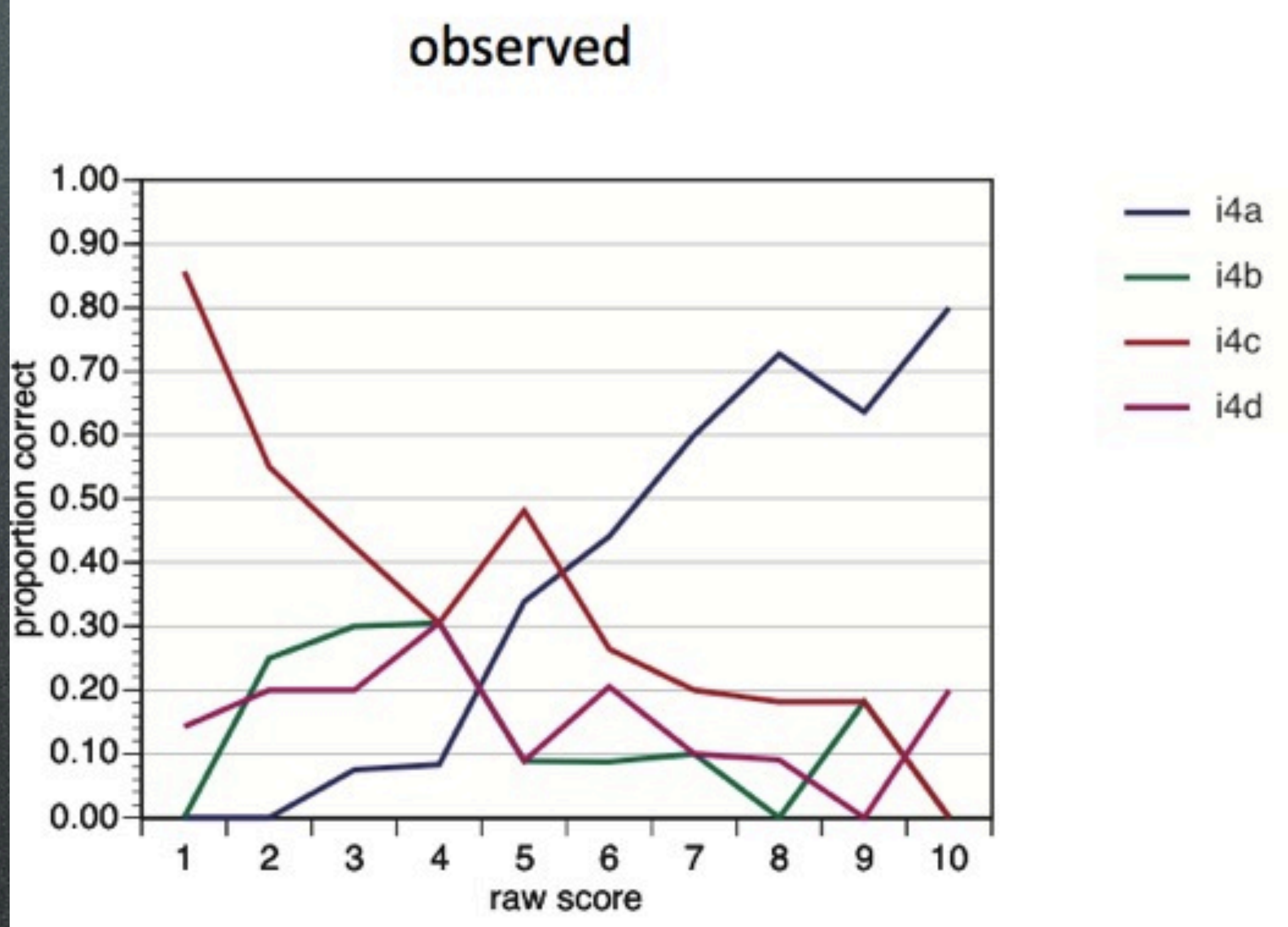

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A Learning Progression, according to Corcoran, Mosher, and Rogat (2009), contains:

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MStar Progressions

LP1: Understanding Positive Rational Numbers, their Representations, and their Uses

LP2: Understanding Variable Expressions, and their Applications

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- Equipartitioning
- Decomposition

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- Meaning of Addition
- Meaning of Multiplication
- Meaning of Division
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LP2

Understanding Variable Expressions, and their Applications

- Variables as Unknown Quantity
- Evaluate
- Verbal Translations of Expressions and Equations
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Example of Sublevels

Equivalent Fractions Progression

	Level Description	Misconceptions

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4.1	**i.** Given a diagram, the student understands that different fractions can represent the same magnitude.	i. Is not able to generate equivalent fractions without being given a diagram.

Example of Sublevels

Equivalent Fractions Progression

	Level Description	Misconceptions
4.1	**i.** Given a diagram, the student understands that different fractions can represent the same magnitude.	i. Is not able to generate equivalent fractions without being given a diagram.
4.2	**i.** Given a diagram, the student can recognize a model that represents an equivalent fraction. The student understands that equivalent fractions will always occupy the same point on the number line. **ii.** The student understands that the number and size of the parts differ even though the two fractions themselves are equivalent. (e.g. 3/4 has 3 "bigger" parts, and 6/8 has 6 "smaller" parts.)	**i.** Cannot generate equivalent fractions, can only recognize equivalence when given the models. When asked if two fractions are equivalent, they make mistakes based on estimating partitions (e.g. conclude that 3/5 and 6/10 are not equivalent because in their drawing the points did not exactly match up) **ii.** Does not recognize when "denominators" are easily related as multiples of each other. (e.g. that denominators or 6 and 12 are easily related; but 3 and 5 are not as easily related.)

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4.3	**i.** The student can generate simple equivalent fractions using a visual model (i.e., area model or number line). **ii.** The student can find common denominators needed to write equivalent fractions i.e. 3/4 as 18/24.	**i.** The student confuses relative equivalence and absolute equivalence. The fractional representation may be equivalent but the value is not equivalent (i.e., 1/4 of a meter is not the same distance as 3/12 of a kilometer). **ii.** Cannot generalize the process that dividing the denominator into "n" equal parts results in a numerator that is exactly "n" times as big.

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4.4	**ii.** The student understands the mathematical reasoning behind generating equivalent fractions (n/n * a/b = a/b), including that a number divided by itself is 1 (n/n = 1), and the identity property of multiplication (n * 1 = n). The student can generalize the dividing the denominator into "n" equal parts results in numerator that is exactly "n" times as big.	

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Interaction of Progress Variables: LP1

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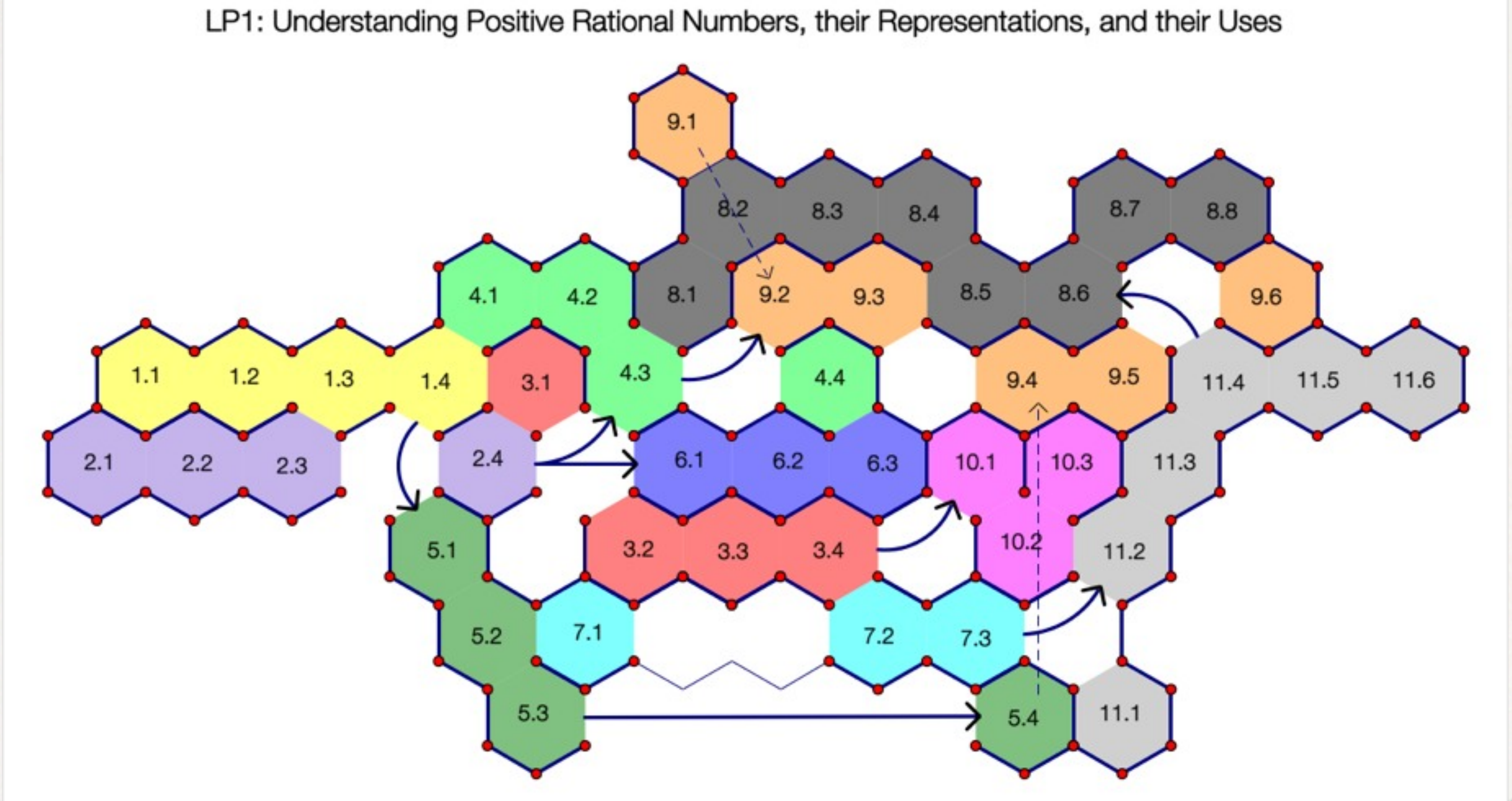
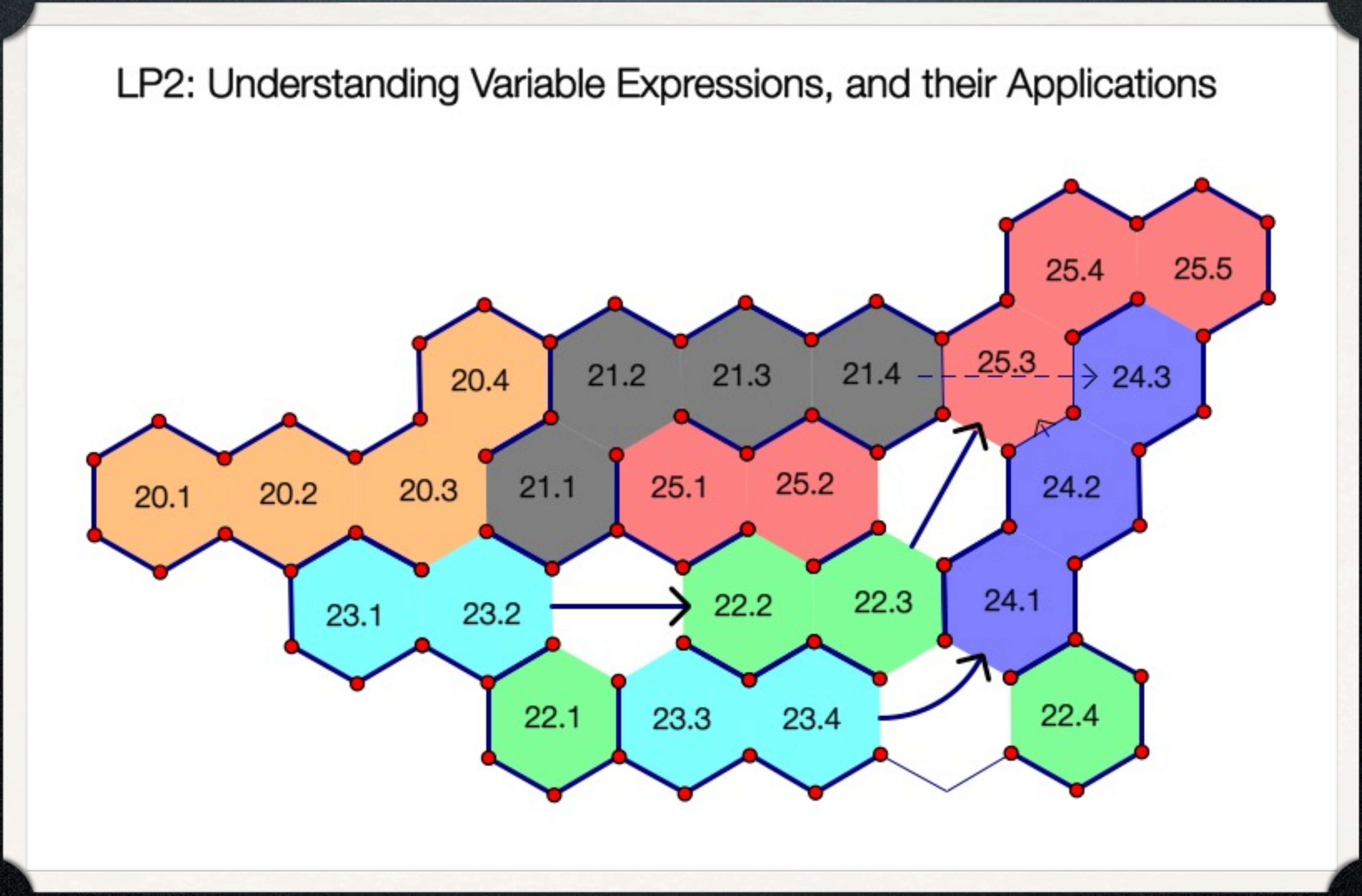


Interaction of Progress Variables: LP2



Validity

- Qualitative analysis from student interviews
- Understanding how these can be used at a "systems" level for content